Analyse de Graphes par Ondelettes

Construction of Tight Frames on Graphs and Application to Denoising

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Introduction

Introduction

Process data efficiently taking into account

- Dimension
- Underlying manifold
- Time, scale localization
- Noise

Basics

Setting

Signal supported on a manifold $\mathscr{M}$

$$\mathscr{D} = \{x_1, ..., x_n\} \subset \mathscr{M} \subset \mathbb{R}^d \tag{1}$$

Noisy signal :

$$yi = f(x_i) + \varepsilon_i \tag{2}$$

Noise ε is i.i.d. and centered

Frame

Let the Hilbert space $\mathscr{H}$, the countable set $\{z_i\}_{i \in I} \subset \mathscr{H}$ is a frame with bounds A, B if :

$$\forall Z \in \mathscr{H} : \|z\| A \leq \sum_{i \in I} |\langle z, z_i \rangle|^2 \leq \|z\| B \tag{3}$$

Parseval $\implies A = B = 1$

Frame operators

Matrix point of view of operators :

- Analysis : $Tz = (\langle z, z_i \rangle)_{i \in I} \in \mathbb{R}^I$
- Synthesis : $T^*a = \sum a_i z_i \in \mathbb{R}^d$
- Frame : TT^*z
- Gramian : T^*Ta

Frame properties

$\{z_i\}$ is a Parseval frame iif :

- $\forall y \in \mathscr{H} : \sum_i \langle y, z_i \rangle z_i$
- Frame operator S in identity on $\mathbb{R}^n$
- Gramian operator U is an orthogonal projector of rank n in $\mathbb{R}^k$

Furthermore, if $\{z_i\}$ is a Parseval frame :

- $\|z_i\| \leq 1$, for $i \in \{1, \cdots, k\}$
- dim $\mathscr{H} = n = \sum_i \|z_i\|^2$

Neighborhood Graphs

On a graph $G = (V, E)$, with weight function w, we note :

$$A_{i,j} = w((v_i, v_j)) \text{ if } (v_i, v_j) \in E \tag{4}$$

$$d_i = \sum_{j=1}^{|A|} A_{i,j} \tag{5}$$

We use the Euclidian distance $d(x_i, x_j) = \|x_i - x_j\|$.
One can choose as Neighborhood Graph :

- k-nearest-neighbor graph (k-NN-graph), possibly weighted
- ε-graph, possibly weighted
- complete graph

Spectral Graph Theory

Unormalized Laplace operator :

$$L^u = D - A \tag{6}$$

Normalized Laplace operator :

$$L^{norm} = I_n - D^{-1/2} A D^{1/2} \tag{7}$$

With $D = diag(d_1, \cdots, d_n)$
The normalized eigenvectors Φ_i of the graph Laplace operator form an orthonormal basis of $\mathbb{R}^n$

Construction and Properties

Frame elements : General characterisation

Use localized elements rather than $\{\Phi_i\}$.
Considering $\zeta_k : \mathbb{R}_+ \to [0, 1]$ s.t.

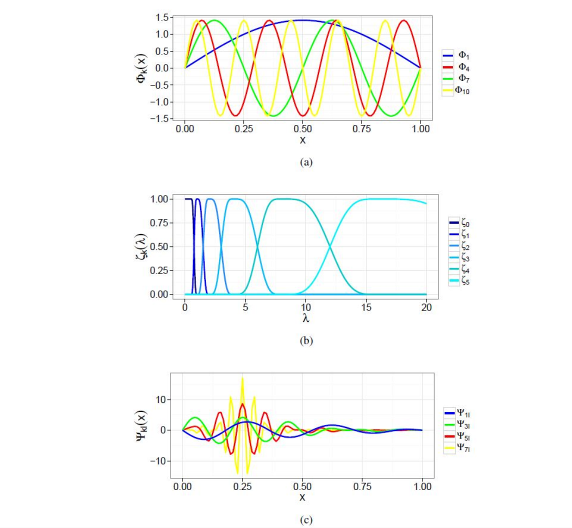
- $\sum_{j \geq 0} \zeta_j(x) = 1$ for all $x \geq 0$
- $\#\{\zeta_k : \zeta_k(\lambda_i) \neq 0\} < \infty$

One may define the following Parseval frame (see Th.2) of functions :

$$\Psi_{kl} = \sum_{i=1}^{n} \sqrt{\zeta_k(\lambda_i)} \Phi_i(x_l) \Phi_i \in \mathbb{R}^n \tag{8}$$

$0 \leq k \leq Q := \max\{k : \exists i, \zeta_k(\lambda_i) > 0\}$
$1 \leq j \leq n$

Frame elements : General characterisation



Multiscale bandpass filter

Let $g \in C^\infty(\mathbb{R}_+)$ s.t.

- supp $g \subset [0, 1]$
- $0 \leq g \leq 1$
- $g(u) = 1$ for $u \in [0, 1/b]$

Let $b > 1$, one may use as ζ_k :

$$\zeta_k(x) := \begin{cases} g(x) \text{ if } k = 0 \\ g(b^{-k}x) - g(b^{-k+1}x) \text{ if } k > 0 \end{cases} \tag{9}$$

$\implies \zeta_k(x) = \zeta_1(b^{-k}x)$ for $k \geq 1$

Spacial localization

Because the set form a Parseval frame :

$$\Psi_{kl}(x) \leq O\left([d(x, x_l)/b^k]^{-v}\right) \forall v > 0 \tag{10}$$

Example of g strongly localized on $[0, 1]$:

$$\Psi_{kl}(y) := \sin(kl)\sin(ky) \tag{11}$$

Denoising

Soft thresholding

$y_i = f(x_i) + \varepsilon_i$, $\mathbb{E}[\varepsilon_i] = 0$, $Var(\varepsilon_i) = \sigma^2$
Coefficients may be written :

$$b_{kl} = \langle \Psi_{kl}, y \rangle \tag{12}$$

Soft thresholding coefficients technique :

$$S_s(z, c) = sgn(z)(|z| - c)_+ \tag{13}$$

Setting a parameter t, we set $c_{kl} = \sigma \|\Psi_{kl}\| t$. Then, we can estimate f :

$$\hat{f}_s = \sum S(b_{kl}, c_{kl})\Psi_{kl} = T^* S(b, c) \tag{14}$$

Performance

Mean Square Error measure (MSE)

$$Risk(\hat{f}, f) = \mathbb{E}_\varepsilon\left(\|\hat{f} - f\|^2\right) \tag{15}$$

Oracle technique case, "keep or kill" :

$$Risk(\hat{f}_I, f) \leq \sum \left(a_{kl}^2.\chi\{(k,l) \notin I\}\right) + \sigma^2\|\Psi_{kl}\|^2.\chi\{(k,l) \in I\} \tag{16}$$

Follows bounded minimization :

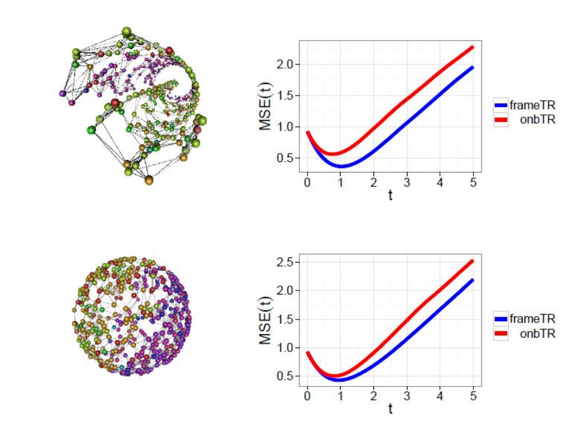
$$\inf_I Risk(\hat{f}_I, f) \leq \sum_{kl} \min\left(\langle f, \Psi_{kl}\rangle, \sigma^2\|\Psi_{kl}\|^2\right) =: OB(f) \tag{17}$$

Performance

$$Risk(\hat{f}, f) \leq (2\log(n) + 1)(\sigma^2 + OB(f)) \qquad (18)$$

Results

Test on piecewise constant functions



Test on piecewise constant functions

Example: sphere, jump function, $\sigma^2 = 1, n = 500, m = 50$

Graph	L	FrTh		LETh		LETr	
kNN	U	0.510	(0.050)	0.693	(0.061)	0.905	(0.108)
kNN	N	0.538	(0.046)	0.712	(0.055)	0.931	(0.094)
WkNN	U	0.521	(0.049)	0.652	(0.050)	0.800	(0.097)
WkNN	N	0.530	(0.049)	0.674	(0.057)	0.749	(0.091)
CGK	U	0.520	(0.055)	0.638	(0.065)	0.821	(0.107)
CGK	N	0.530	(0.052)	0.670	(0.050)	0.725	(0.081)
εG	U	0.505	(0.058)	0.650	(0.068)	0.865	(0.115)
εG	N	0.557	(0.052)	0.710	(0.059)	0.902	(0.106)
WεG	U	0.482	(0.055)	0.622	(0.064)	0.787	(0.111)
WεG	N	0.530	(0.049)	0.674	(0.057)	0.749	(0.091)

Smoothing Kernel Regression: min. MSE = 0.612 (0.066)
Kernel Ridge Regression: min. MSE = 0.594 (0.051)

Example: swiss roll, jump function, $\sigma^2 = 1, n = 500, m = 50$

Graph	L	FrTh		LETh		LETr	
kNN	U	0.462	(0.043)	0.647	(0.039)	0.876	(0.079)
kNN	N	0.494	(0.043)	0.676	(0.043)	0.902	(0.071)
WkNN	U	0.443	(0.045)	0.600	(0.050)	0.790	(0.102)
WkNN	N	0.500	(0.043)	0.659	(0.045)	0.775	(0.079)
CGK	U	0.491	(0.053)	0.625	(0.057)	0.844	(0.096)
CGK	N	0.520	(0.047)	0.648	(0.049)	0.713	(0.079)
εG	U	0.459	(0.049)	0.610	(0.053)	0.872	(0.095)
εG	N	0.532	(0.045)	0.681	(0.050)	0.884	(0.089)
WεG	U	0.441	(0.049)	0.574	(0.049)	0.793	(0.113)
WεG	N	0.503	(0.045)	0.643	(0.051)	0.744	(0.089)

Smoothing Kernel Regression: min. MSE = 0.589 (0.082)
Kernel Ridge Regression: min. MSE = 0.779 (0.052)

Ending

Reference :

Göbel, F., Blanchard, G. and von Luxburg, U., 2014. Construction of tight frames on graphs and application to denoising. *arXiv preprint arXiv* :1408.4012.

End

Thank you for your attention.