

Paper summary of: Sevi, Rilling, and Borgnat  
“Harmonic analysis on directed graphs and  
applications: from Fourier analysis to wavelets”,  
[SRB18]

Destouet, Gabriel

June 24, 2020

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Filters are defined with  $\{\alpha_k\}_k$ , wavelets by dilation  $s$  of  $h(sL)$

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  - It gives a Laplacian for directed graphs
  - and leads to a frequency interpretation of the spectral properties of  $A$

# From Adjacent Matrix to Random Walk Operator

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$$\pi P(y) = \sum \pi(x)p(x, y) \quad (5)$$

## Some properties of the Random Walk Operator

- It is **irreducible** if:

$$\forall x, y \in \mathcal{V}, \exists m < \infty : \mathbb{P}(X_{n+m} = y | x_n = x) > 0 \quad (6)$$

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<sup>1</sup>Actually if  $P$  is irreducible, we almost have the same properties, see

## Relation between $P$ and $P^*$

In [SRB18] they require that  $P$  is *ergodic* to have

$$p^*(x, y) = \frac{\pi(y)}{\pi(x)} p(y, x) \quad (9)$$

$$\Leftrightarrow P^* = \Pi^{-1} P^T \Pi \quad (10)$$

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- ▶ Actually,  $\pi$  can also be estimated if  $P$  is only *irreducible*, see [Mey00, Chap.8]

## Two transformations of $P$

With  $P$  *irreducible*

- To make  $P$  *aperiodic* (and thus *ergodic*)

$$\tilde{P} = \{ \tilde{P}_\gamma : \tilde{P}_\gamma = (1 - \gamma)P + \gamma I \mid \gamma \in [0, 1] \} \quad (12)$$

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- Set of convex combination of  $P$  and  $P^*$

$$\bar{P} = \{ \bar{P}_\alpha : \bar{P}_\alpha = (1 - \alpha)P + \alpha P^* \mid \alpha \in [0, 1] \} \quad (13)$$

Only  $\bar{P}_{1/2} = \bar{P}$  is reversible.

## Two function spaces

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- ▶  $\bar{P}$  is self adjoint and thus has an orthonormal eigenbasis.

Laplacian for directed graphs in  $\ell^2(\mathcal{V})$  and  $\ell^2(\mathcal{V}, \pi)$

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$\mathcal{L} = \psi^{-1} \mathcal{L}_{RW} \psi$ .  $\mathcal{L}_{RW}$  and  $\bar{P}$  have the same (orthonormal) eigenspace.

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- ▶ Dirichlet energy of a graph signal  $f \in \ell^2(\mathcal{V}, \pi)$  on  $P$

(19)

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For each  $(\xi, \nu)$  of  $P$  we are able to associate a frequency  $\omega = 1 - \Re(\nu) \in [0, 2]$

## Fourier Analysis on finite groups

- Example of the classical circulant matrix where  $P = C_N$

$$C_N = \begin{pmatrix} 0 & 1 & \cdots & \cdots & 0 \\ \vdots & 0 & 1 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 1 \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

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- ▶ Example of the classical circulant matrix where  $P = C_N$
- ▶ The authors [SRB18] consider a transformation  $\tilde{P}_\gamma$  of  $P$  in order to have an irreducible and aperiodic (ergodic) operator and find:

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- ▶ Extension to the case of toroidal graph  $\mathcal{T}_{m,n} = \mathcal{C}_m \square \mathcal{C}_n$  where  $\mathcal{C}_m, \mathcal{C}_n$  are directed cycle graphs.

## Graph filters with the Random Walk Operator

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- ▶ Graph Filter  $H$  as a linear combination of spectral projectors  $\mathbf{E}_{\nu_k}$  associated with eigenvalues  $\nu_k$

$$H_{\omega} = \sum_k \gamma_k \mathbf{E}_{\nu_k} \quad (21)$$

$$= \sum_{\omega \in \omega} \tau_{\omega} S_{\omega} \quad (22)$$

Where

$$S_{\omega} = \sum_{\nu : \omega = 1 - \Re(\nu)} \mathbf{E}_{\nu}$$

By defining  $h : \omega \rightarrow \mathbb{R}(\mathbb{C})$ , we have the graph filter with frequency response

$$H = \sum_{\omega \in \omega} h(\omega) S_{\omega}$$

## Multiresolution analysis on directed graph

- Bank of synthesis  $\mathcal{K}$  and analysis  $\tilde{\mathcal{K}}$  defined as :

$$\mathcal{K} = \{H_{t_J}, G_{t_1}, \dots, G_{t_J}\} \quad (23)$$

$$\tilde{\mathcal{K}} = \{\tilde{H}_{t_J}, \tilde{G}_{t_1}, \dots, \tilde{G}_{t_J}\} \quad (24)$$

Where

$$H_t = \sum_k h(t\omega_k) S_k \text{ where } h \text{ is a low pass} \quad (25)$$

$$G_t = \sum_k g(t\omega_k) S_k \text{ where } g \text{ is a high pass} \quad (26)$$

With  $S_k$  the random walk spectral projectors previously defined associated to mono-frequencies  $\omega_k$

- Wavelets:  $h_{t_J,k} = H_{t_J} \delta_k$  and  $g_{t_j,k} = G_{t_j} \delta_k$

## Critically sampled wavelets

- Use the diffusion operator  $T = \Pi^{1/2} P \Pi^{-1/2}$  to find the bases (scaling functions)  $\{\Phi_j\}_{1 \leq j \leq J}$  of nested spaces  $\{V_j\}_{1 \leq j \leq J}$ ,  $V_J \subset V_{J-1} \cdots \subset V_0$

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- ▶ Get a set of scaling functions  $\Phi_j$  spanning spaces  $V_j$ .
- ▶ Diffusion wavelets  $\Psi_j$  are obtained as the bases of the complement  $W_j$  of  $V_{j+1}$  in  $V_j$

$$\Psi_j = \Phi_j - \Phi_{j+1} \Phi_j^t \Phi_{j+1}$$

## Some wavelets and scaling functions with diffusion method on cycle graph

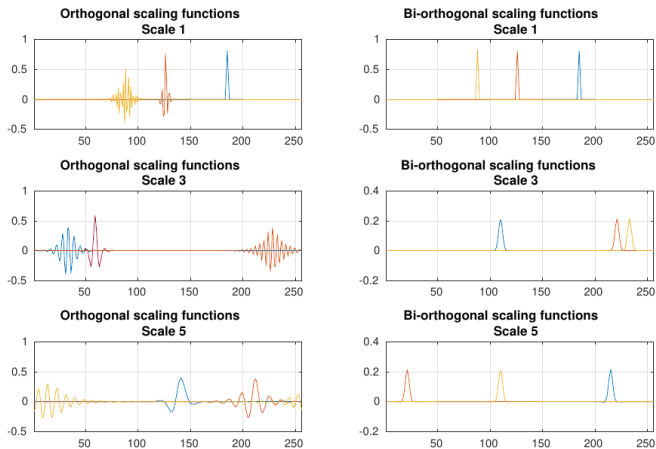


FIGURE 10. Orthogonal and biorthogonal scaling functions on the directed cycle graph  $\mathcal{C}_{256}$ .

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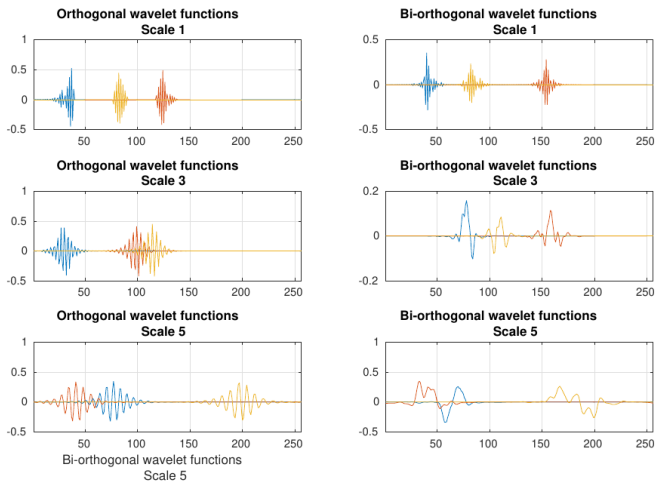


FIGURE 11. Orthogonal and biorthogonal wavelet functions on the directed cycle graph  $\mathcal{C}_{256}$ .

## Comparison between scaling functions from Graph Filters and from diffusion

- ▶ Case 2: Use the spectral properties of  $\bar{T}_\alpha = \Pi^{1/2} \bar{P}_\alpha \Pi^{-1/2}$  to define:

$$H_\alpha = \sum_{\omega \in \omega} h(t\omega) S_{w,\alpha}$$

With  $t = 2^4$  and  $h(x) = \exp(-x)$ .

- ▶ Case 3: For different scales  $j$ :  $\{T^{2^j}\}_{j=1}^5$  for *diffusion wavelet* and  $\{\bar{T}^{2^j}\}_{j=1}^5$  for *spectral wavelets*

# Comparison between scaling functions from Graph Filters and from diffusion

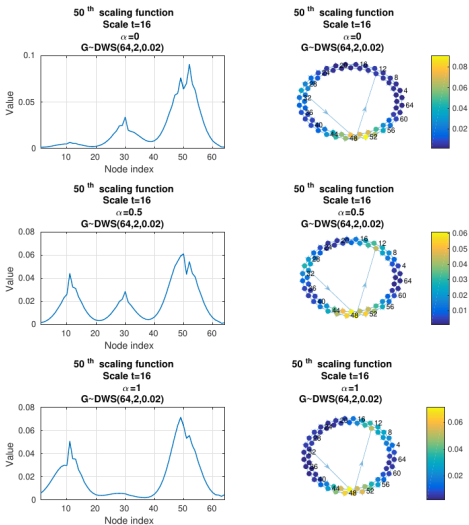


FIGURE 13. 50<sup>th</sup> scaling function at scale 4 on a graph  $\mathcal{G} \sim \text{DWS}(64, 2, 0.02)$ ,  $\alpha \in \{0, 0.5, 1\}$ , eq. (33).

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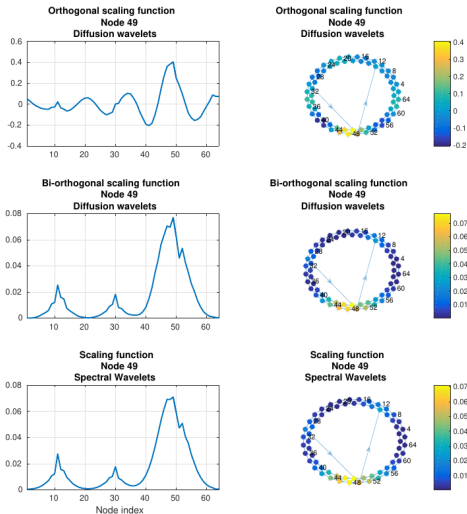


FIGURE 14. Orthogonal and bi-orthogonal scaling functions built w.r.t the diffusion wavelet framework versus scaling function built w.r.t spectral wavelets framework.

## Semi-supervised Learning

- ▶ Method 1: for  $y$  in  $\ell^2(\mathcal{V})$

$$\operatorname{argmin}_f c \|M_I(f - y)\|^2 + c \|(I - M_I)f\|^2 + \rho_2 \langle f, \mathcal{L}f \rangle$$

Where  $M_I$  is the diagonal matrix with 0 on vertices with unknown labels

- ▶ Method 2: for  $y$  in  $\ell^2(\mathcal{V}, \pi)$

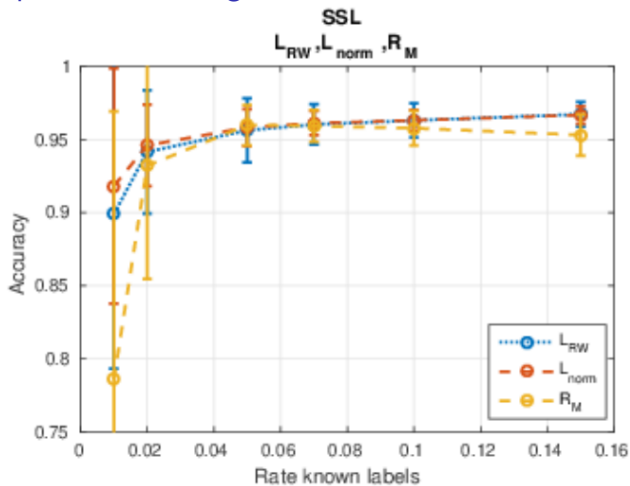
$$\operatorname{argmin}_f c \|M_I(f - y)\|_\pi^2 + c \|(I - M_I)f\|_\pi^2 + \rho_2 \langle f, \mathcal{L}_{RW}f \rangle_\pi$$

- ▶ Method 3: baseline method from [SM13]

$$\operatorname{argmin}_f c \|M_I(f - y)\|^2 + c \|f - W^{\text{norm}}f\|^2$$

- ▶ Benchmark: graph of political blogs with binary label  $-1/1$

## Semi-supervised Learning



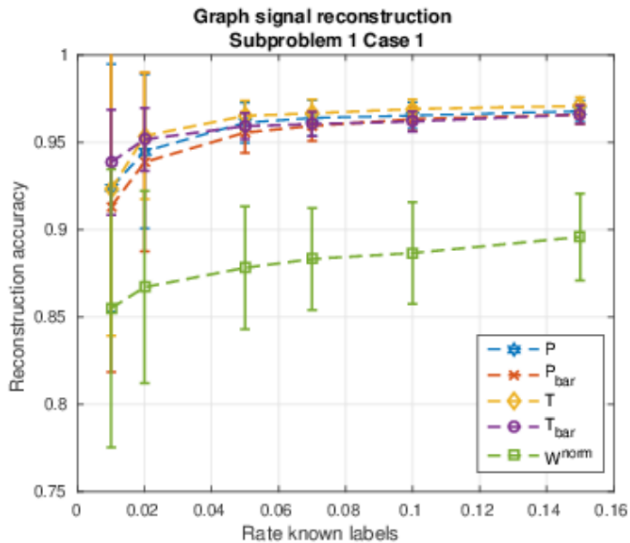
## Graph Signal Reconstruction

- ▶ Random graph signal  $y$  with missing values. Objective:

$$\operatorname{argmin}_{\theta=\{\theta_k\}} \mathbb{E}[\|f_0 - \sum_k \theta_k R^k y\|^2]$$

- ▶ Results with different reference operator  
 $R \in \{P, \bar{P}, T, \bar{T}, W^{\text{norm}}\}$

## Graph Signal Reconstruction



## Conclusion

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- ▶ Good results on graph signal reconstruction and semi-supervised learning.
- ▶ The ergodic constraint on  $P$  might not be necessary, maybe irreducible is sufficient.



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