Paper summary of: Sevi, Rilling, and Borgnat
“Harmonic analysis on directed graphs and applications: from Fourier analysis to wavelets”, [SRB18]

Destouet, Gabriel

June 24, 2020
Graph Signal Processing on undirected graphs

The adjacency matrix $A$ is real symmetric. Thus Laplacians ($L = D - A$, $L_n$, $L_d$, ... ) have good spectral properties — diagonalizable with orthonormal basis and eigenvalues $v_i \in \mathbb{R}^+$ and $v_i \leftrightarrow \omega$ frequency.

Filters or wavelets $H$ as a finite polynomial sum of a reference operator ($e.g.$ $R = L$):

$$H = h(L) = \sum_k \alpha_k L^k$$

Filters are defined with $\{\alpha_k\}$, wavelets by dilation $s$ of $h(sL)$.
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This paper: Graph Signal Processing on \textit{directed} graphs

The adjacency matrix $A$ is real non symmetric

"Naive" Laplacians ($L = D - A$, $L_n$, $L_d$, ...) for undirected graphs are generally not adapted – if diagonalizable, eigenvalues $\nu_i \in \mathbb{C}$ and $\nu_i \leftrightarrow ?$

In general: we are interested in operators which measure the smoothness of signals $f$ and their spectral properties leads to Fourier-like basis.

The questions are:

1. What type of Laplacian should we use to measure the smoothness of signals on directed graphs?

2. What reference operator can we use to define filters on directed graphs?

The authors of [SRB18] propose to use the random walk operator:

- It gives a Laplacian for directed graphs
- And leads to a frequency interpretation of the spectral properties of $A$
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From Adjacent Matrix to Random Walk Operator

Given a graph $G = (V, E, W)$, the Random Walk Operator is

$$P = D^{-1}W$$

Where $D^{-1}$ is the diagonal matrix of out-degrees of $W$, $D_i = \sum w_{i,j}$.

$P$ is the transition matrix of the Markov Chain defined on $G$

$$p(x, y) = P_x, y = P(x_{n+1} = y | x_n = x)$$

Operations with the Random Walk Operator

Left and right operations of $P$:

$$Pf(x) = \sum p(x, y)f(y) = E_P[y | X = f]$$

mass averaged and propagated back to child node $x$

$$\pi P(y) = \sum \pi(x) p(x, y)$$

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$$ \pi P(y) = \sum \pi(x) p(x, y) $$

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mass averaged and propagated back to child node $x$
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Some properties of the Random Walk Operator

- It is **irreducible** if:

  \[ \forall x, y \in \mathcal{V}, \exists m < \infty : P(X_{n+m} = y | X_n = x) > 0 \]  \hspace{1cm} (6)

This is equivalent to say that \( \mathcal{G} \) is strongly connected.

- It is **aperiodic** if:

  \[ \forall x \in \mathcal{V}, \gcd\{n \in \mathbb{N}^+: P(X_{m+n} = x | X_m = x) > 0\} = 1 \]  \hspace{1cm} (7)

- It is **ergodic** if aperiodic and irreducible.

- It is **reversible** if

  \[ P^*x, y = P(x, y) = P(X_n = y | X_{n+1} = x) \]  \hspace{1cm} (8)

  \[ \text{Actually if } P \text{ is irreducible, we almost have the same properties, see } [Mey00, \text{Chap.8}] \]
Some properties of the Random Walk Operator

- **Irreducible** if:

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- If $P$ is ergodic, $P$ has a single eigenvalue $\lambda_{\text{max}} = 1$ and
  $\{\forall \lambda \neq 1, |\lambda| < 1\}$ (Perron-Frobenius Theorem)
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Relation between $P$ and $P^*$

In [SRB18] they require that $P$ is *ergodic* to have

$$p^*(x, y) = \frac{\pi(y)}{\pi(x)} p(y, x)$$  \hspace{1cm} (9)$$

$$\Leftrightarrow \quad P^* = \Pi^{-1} P^T \Pi$$  \hspace{1cm} (10)$$

Where $\Pi = \text{diag}(\pi(v_1), \ldots, \pi(v_N))$, $v_i \in \mathcal{V}$
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- Maybe the authors of [SRB18] need **ergodicity** to estimate $\pi$ via power iteration method or MCMC methods since:

$$P^n(x, .) \xrightarrow[n\to\infty]{} \pi(x)$$

When $P$ is **ergodic**.
Relation between $P$ and $P^*$
In [SRB18] they require that $P$ is *ergodic* to have

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$$P^n(x, .) \xrightarrow[n \to \infty]{} \pi(x) \quad (11)$$

When $P$ is *ergodic*.
- Actually, $\pi$ can also be estimated if $P$ is only *irreducible*, see [Mey00, Chap.8]
Two transformations of $P$

With $P$ irreducible

- To make $P$ aperiodic (and thus ergodic)

$$\tilde{P} = \{ \tilde{P}_\gamma : \tilde{P}_\gamma = (1 - \gamma)P + \gamma I \mid \gamma \in [0, 1] \} \quad (12)$$

$\tilde{P}$ has the same eigenspace than $P$ but is *aperiodic*
Two transformations of $P$
With $P$ irreducible

▸ To make $P$ aperiodic (and thus ergodic)

$$\tilde{P} = \{ \tilde{P}_\gamma : \tilde{P}_\gamma = (1 - \gamma)P + \gamma I \mid \gamma \in [0, 1] \}$$ \hspace{1cm} (12)

$\tilde{P}$ has the same eigenspace than $P$ but is aperiodic

▸ Set of convex combination of $P$ and $P^*$

$$\bar{P} = \{ \bar{P}_\alpha : \bar{P}_\alpha = (1 - \alpha)P + \alpha P^* \mid \alpha \in [0, 1] \}$$ \hspace{1cm} (13)

Only $\bar{P}_{1/2} = \bar{P}$ is reversible.
Two function spaces

- On $G$, space of graph signals $f, g \in \ell^2(V)$ with

$$\langle f, g \rangle = \sum_{x \in V} f(x)\bar{g}(x)$$
Two function spaces

- On $G$, space of graph signals $f, g \in \ell^2(\mathcal{V})$ with
  \[
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  \]

- On $G$ with $\pi$, space of graph signals $f, g \in \ell^2(\mathcal{V}, \pi)$ with
  \[
  \langle f, g \rangle_{\pi} = \sum_{x \in \mathcal{V}} f(x)\bar{g}(x)\pi(x)
  \]
Two function spaces

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- On $G$ with $\pi$, space of graph signals $f, g \in \ell^2(\mathcal{V}, \pi)$ with
  $$\langle f , g \rangle_{\pi} = \sum_{x \in \mathcal{V}} f(x) \bar{g}(x) \pi(x)$$

- We have an isometry $\psi$ between $\ell^2(\mathcal{V})$ and $\ell^2(\mathcal{V}, \pi)$
  $$\forall f \in \ell^2(\mathcal{V}) , \quad \psi : f \mapsto \Pi^{-1/2} f$$

$\bar{f}$ is self adjoint and thus has an orthonormal eigenbasis.
Two function spaces

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- In particular, $P$ and $P^*$ are adjoint in $\ell^2(\mathcal{V}, \pi)$:
  $\langle f, Pg \rangle_\pi = \langle P^*f, g \rangle_\pi$  (14)
Two function spaces

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- On $G$ with $\pi$, space of graph signals $f, g \in \ell^2(\mathcal{V}, \pi)$ with
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- In particular, $P$ and $P^*$ are adjoint in $\ell^2(\mathcal{V}, \pi)$:
  \[ \langle f, Pg \rangle_\pi = \langle P^* f, g \rangle_\pi \] (14)

- $\bar{P}$ is self adjoint and thus has an 
Laplacian for directed graphs in $\ell^2(V)$ and $\ell^2(V, \pi)$
Laplacian for directed graphs in $\ell^2(V)$ and $\ell^2(V, \pi)$

- First introduce $T$

$$T = \Pi^{1/2} P \Pi^{-1/2}$$

Which is equivalent to $\psi^{-1} P \psi$
Laplacian for directed graphs in $\ell^2(\mathcal{V})$ and $\ell^2(\mathcal{V}, \pi)$

- First introduce $T$

$$T = \Pi^{1/2}P\Pi^{-1/2}$$

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- Let $\mathcal{L}$ in $\ell^2(\mathcal{V})$

$$\mathcal{L} = I - \frac{T + T^T}{2}$$ (15)
Laplacian for directed graphs in $\ell^2(\mathcal{V})$ and $\ell^2(\mathcal{V}, \pi)$

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- Let $\mathcal{L}$ in $\ell^2(\mathcal{V})$

$$\mathcal{L} = I - \frac{T + T^T}{2} \quad (15)$$

- Let $\mathcal{L}_{RW}$ in $\ell^2(\mathcal{V}, \pi)$

$$\mathcal{L}_{RW} = I - \bar{P} \quad (16)$$
Laplacian for directed graphs in $\ell^2(\mathcal{V})$ and $\ell^2(\mathcal{V}, \pi)$

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- Let $\mathcal{L}_{RW}$ in $\ell^2(\mathcal{V}, \pi)$
  
  $$\mathcal{L}_{RW} = I - \bar{P} \quad (16)$$

$$\mathcal{L} = \psi^{-1} \mathcal{L}_{RW} \psi.$$  $\mathcal{L}_{RW}$ and $\bar{P}$ have the same (orthonormal) eigenspace.
The Dirichlet energy and its link with $\mathcal{L}_{RW}$

\[
\begin{align*}
\text{Dirichlet energy of a graph signal } f \in \ell^2(V, \pi) \text{ on } P_D^2 \pi, P(f) &= \frac{1}{2} \sum_{(x, y) \in E} \pi(x) p(x, y) |f(x) - f(y)|^2 \\
&= \langle f, \mathcal{L}_{RW} f \rangle \pi \\
R_{\pi, P}(f) &= D_{\pi, P} f \|f\|_{\pi}^2
\end{align*}
\]
The Dirichlet energy and its link with $\mathcal{L}_{RW}$

- Dirichlet energy of a graph signal $f \in \ell^2(\mathcal{V}, \pi)$ on $P$

(19)
The Dirichlet energy and its link with $\mathcal{L}_{RW}$

- Dirichlet energy of a graph signal $f \in \ell^2(V, \pi)$ on $P$

\[
\mathcal{D}^2_{\pi, P}(f) = \frac{1}{2} \sum_{(x,y) \in \mathcal{E}} \pi(x)p(x, y)|f(x) - f(y)|^2
\]  

(17)

\[
R_{\pi, P}(\xi) = 1 - \Re(\nu)
\]

(20)

For each $(\xi, \nu)$ of $P$ we are able to associate a frequency \( \omega = 1 - \Re(\nu) \in [0, 2] \).
The Dirichlet energy and its link with $\mathcal{L}_{RW}$

- Dirichlet energy of a graph signal $f \in \ell^2(\mathcal{V}, \pi)$ on $P$

$$\mathcal{D}_{\pi,P}^2(f) = \frac{1}{2} \sum_{(x,y) \in \mathcal{E}} \pi(x)p(x,y)|f(x) - f(y)|^2$$  \hspace{1cm} (17)$$

$$= \langle f, \mathcal{L}_{RW}f \rangle_\pi$$ \hspace{1cm} (18)

Where $R_{\pi,P}$ is the Rayleigh quotient.

- For any eigenvectors $\xi$ of $P$ with eigenvalue $\nu$.

$$R_{\pi,P}(\xi) = 1 - \Re(\nu)$$ \hspace{1cm} (20)

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$$R_{\pi,P}(\xi) = 1 - \Re(\nu)$$ \hspace{1cm} (20)
The Dirichlet energy and its link with $\mathcal{L}_{RW}$

- Dirichlet energy of a graph signal $f \in \ell^2(\mathcal{V}, \pi)$ on $P$

$$D^2_{\pi,P}(f) = \frac{1}{2} \sum_{(x,y) \in \mathcal{E}} \pi(x)p(x,y)|f(x) - f(y)|^2$$  \hspace{1cm} (17)

$$= \langle f, \mathcal{L}_{RW}f \rangle_\pi$$  \hspace{1cm} (18)

$$\mathcal{R}_{\pi,P}(f) = \frac{D^2_{\pi,P}f}{\|f\|^2_\pi}$$  \hspace{1cm} (19)

Where $\mathcal{R}_{\pi,P}$ is the Rayleigh quotient.
The Dirichlet energy and its link with $\mathcal{L}_{RW}$

- Dirichlet energy of a graph signal $f \in \ell^2(\mathcal{V}, \pi)$ on $P$

\[
\mathcal{D}^2_{\pi,P}(f) = \frac{1}{2} \sum_{(x,y) \in \mathcal{E}} \pi(x)p(x,y)|f(x) - f(y)|^2
\]

\[
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$$= \langle f, \mathcal{L}_{RW}f \rangle_\pi$$

$$\mathcal{R}_{\pi,P}(f) = \frac{D_{\pi,P}^2f}{\|f\|_{2\pi}^2}$$

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$$\mathcal{R}_{\pi,P}(\xi) = 1 - \Re(\nu)$$

For each $(\xi, \nu)$ of $P$ we are able to associate a frequency

$$\omega = 1 - \Re(\nu) \in [0, 2]$$
Fourier Analysis on finite groups

- Example of the classical circulant matrix where $P = C_N$

$$C_N = \begin{pmatrix}
0 & 1 & \cdots & \cdots & 0 \\
\vdots & 0 & 1 & \cdots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \vdots & 1 \\
1 & 0 & \cdots & \cdots & 0
\end{pmatrix}$$

- The authors [SRB18] consider a transformation $\tilde{P}_\gamma$ of $P$ in order to have an irreducible and aperiodic (ergodic) operator and find:

$$R_\pi, P(\xi_k) = (1 - \gamma)(1 - \cos(\frac{2\pi (k-1)}{N})) = \omega_k$$

- By ordering the frequencies with the eigenvectors, we retrieve the classical results of signal processing.

- By taking the limit $\gamma \to 0$, they could also define frequency for $P$.

- Extension to the case of toroidal graph $T_{m,n} = C_m C_n$ where $C_m, C_n$ are directed cycle graphs.
Fourier Analysis on finite groups

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Graph filters with the Random Walk Operator
As a reference operator, choose $R = P$
Graph filters with the Random Walk Operator

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- Graph Filter $H$ as a polynomial sum of $P$

$$H = \sum_{k} \theta_k P^k$$
Graph filters with the Random Walk Operator

As a reference operator, choose $R = P$

- Graph Filter $H$ as a linear combination of spectral projectors $E_{\nu_k}$ associated with eigenvalues $\nu_k$

\[
H_\omega = \sum_k \gamma_k E_{\nu_k} \quad (21)
\]

\[
= \sum_{\omega \in \omega} \tau_\omega S_\omega \quad (22)
\]

Where

\[
S_\omega = \sum_{\nu : \omega = 1 - \Re(\nu)} E_\nu
\]

By defining $h : \omega \to \mathbb{R}(\mathbb{C})$, we have the graph filter with frequency response

\[
H = \sum_{\omega \in \omega} h(\omega) S_\omega
\]
Multiresolution analysis on directed graph

- Bank of synthesis $\mathcal{K}$ and analysis $\tilde{\mathcal{K}}$ defined as:

$$\mathcal{K} = \{ H_{t_j}, G_{t_1}, \ldots, G_{t_j} \} \quad (23)$$

$$\tilde{\mathcal{K}} = \{ \tilde{H}_{t_j}, \tilde{G}_{t_1}, \ldots, \tilde{G}_{t_j} \} \quad (24)$$

Where

$$H_t = \sum_k h(t\omega_k)S_k \text{ where } h \text{ is a low pass} \quad (25)$$

$$G_t = \sum_k g(t\omega_k)S_k \text{ where } g \text{ is a high pass} \quad (26)$$

With $S_k$ the random walk spectral projectors previously defined associated to mono-frequencies $\omega_k$

- Wavelets: $h_{t_j,k} = H_{t_j}\delta_k$ and $g_{t_j,k} = G_{t_j}\delta_k$
Critically sampled wavelets

- Use the diffusion operator $T = \Pi^{1/2} P \Pi^{-1/2}$ to find the bases (scaling functions) $\{\Phi_j\}_{1 \leq j \leq J}$ of nested spaces $\{V_j\}_{1 \leq j \leq J}$, $V_J \subset V_{J-1} \cdots \subset V_0$
Critically sampled wavelets

- Use the diffusion operator $T = \Pi^{1/2}P\Pi^{-1/2}$ to find the bases (scaling functions) $\{\Phi_j\}_{1 \leq j \leq J}$ of nested spaces $\{V_j\}_{1 \leq j \leq J}$, $V_J \subset V_{J-1} \cdots \subset V_0$

- Sketch of the algorithm:
Critically sampled wavelets

- Use the diffusion operator $T = \prod^{1/2} P \prod^{-1/2}$ to find the bases (scaling functions) $\{\Phi_j\}_{1 \leq j \leq J}$ of nested spaces $\{V_j\}_{1 \leq j \leq J}, V_J \subset V_{J-1} \cdots \subset V_0$

- Sketch of the algorithm:
  1. Start with: $\Phi_0 = \{\delta_k\}_k, \ p = 2^0$
Critically sampled wavelets

- Use the diffusion operator \( T = \Pi^{1/2} P \Pi^{-1/2} \) to find the bases (scaling functions) \( \{ \Phi_j \}_{1 \leq j \leq J} \) of nested spaces \( \{ V_j \}_{1 \leq j \leq J} \), \( V_J \subset V_{J-1} \cdots \subset V_0 \)

- Sketch of the algorithm:
  1. Start with: \( \Phi_0 = \{ \delta_k \}_k, p = 2^0 \)
  2. Compute \( \tilde{\Phi}_1 = T^p \Phi_0 \)

- Get a set of scaling functions \( \Phi_j \) spanning spaces \( V_j \).
- Diffusion wavelets \( \Psi_j \) are obtained as the bases of the complement \( W_j \) of \( V_j + 1 \) in \( V_j \)
  \[ \Psi_j = \Phi_j - \Phi_{j+1} \]
Critically sampled wavelets

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- Sketch of the algorithm:
  1. Start with: $\Phi_0 = \{\delta_k\}_k$, $p = 2^0$
  2. Compute $\tilde{\Phi}_1 = T^p\Phi_0$
  3. Prune $\tilde{\Phi}_1$ to obtain $\Phi_1$ such that $\|\tilde{\Phi}_1 - \Phi_1\|_F$ is minimal

Diffusion wavelets $\Psi_j$ are obtained as the bases of the complement $W_j$ of $V_j + 1$ in $V_j$ $\Psi_j = \Phi_j - \Phi_{j+1}$
Critically sampled wavelets

- Use the diffusion operator $T = \Pi^{1/2} P \Pi^{-1/2}$ to find the bases (scaling functions) $\{\Phi_j\}_{1 \leq j \leq J}$ of nested spaces $\{V_j\}_{1 \leq j \leq J}$, $V_J \subset V_{J-1} \cdots \subset V_0$

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  2. Compute $\tilde{\Phi}_1 = T^p \Phi_0$
  3. Prune $\tilde{\Phi}_1$ to obtain $\Phi_1$ such that $\|\tilde{\Phi}_1 - \Phi_1\|_F$ is minimal
  4. Update $p \leftarrow 2 \times p$
Critically sampled wavelets

- Use the diffusion operator $T = \Pi^{1/2} P \Pi^{-1/2}$ to find the bases (scaling functions) $\{\Phi_j\}_{1 \leq j \leq J}$ of nested spaces $\{V_j\}_{1 \leq j \leq J}$, $V_J \subset V_{J-1} \cdots \subset V_0$

- Sketch of the algorithm:
  1. Start with: $\Phi_0 = \{\delta_k\}_k$, $p = 2^0$
  2. Compute $\tilde{\Phi}_1 = T^p \Phi_0$
  3. Prune $\tilde{\Phi}_1$ to obtain $\Phi_1$ such that $\|\tilde{\Phi}_1 - \Phi_1\|_F$ is minimal
  4. Update $p \leftarrow 2 \times p$
  5. Compute $\tilde{\Phi}_2 = T^p \Phi_1$

- Get a set of scaling functions $\Phi_j$ spanning spaces $V_j$. Diffusion wavelets $\Psi_j$ are obtained as the bases of the complement $W_j$ of $V_j + 1$ in $V_j$: $\Psi_j = \Phi_j - \Phi_j + 1 \Phi_j$.
Critically sampled wavelets

- Use the diffusion operator $T = \Pi^{1/2}R\Pi^{-1/2}$ to find the bases (scaling functions) $\{\Phi_j\}_{1 \leq j \leq J}$ of nested spaces $\{V_j\}_{1 \leq j \leq J}$, $V_J \subset V_{J-1} \cdots \subset V_0$

- Sketch of the algorithm:
  1. Start with: $\Phi_0 = \{\delta_k\}_k$, $p = 2^0$
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  3. Prune $\tilde{\Phi}_1$ to obtain $\Phi_1$ such that $\|\tilde{\Phi}_1 - \Phi_1\|_F$ is minimal
  4. Update $p \leftarrow 2 * p$
  5. Compute $\tilde{\Phi}_2 = T^p\Phi_1$
  6. Prune $\tilde{\Phi}_2$ to obtain $\Phi_2$ such that $\|\tilde{\Phi}_2 - \Phi_2\|_F$ is minimal

- Get a set of scaling functions $\Phi_j$ spanning spaces $V_j$.
- Diffusion wavelets $\Psi_j$ are obtained as the bases of the complement $W_j$ of $V_j + 1$ in $V_j$

$$\Psi_j = \Phi_j - \Phi_{j+1} \tilde{\Phi}_j \Phi_{j+1}$$
Critically sampled wavelets

- Use the diffusion operator $T = \Pi^{1/2} P \Pi^{-1/2}$ to find the bases (scaling functions) $\{\Phi_j\}_{1 \leq j \leq J}$ of nested spaces $\{V_j\}_{1 \leq j \leq J}$, $V_J \subset V_{J-1} \cdots \subset V_0$

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  5. Compute $\tilde{\Phi}_2 = T^p \Phi_1$
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  :
Critically sampled wavelets

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- Sketch of the algorithm:
  
  1. Start with: $\Phi_0 = \{\delta_k\}_k$, $p = 2^0$
  2. Compute $\tilde{\Phi}_1 = T^p\Phi_0$
  3. Prune $\tilde{\Phi}_1$ to obtain $\Phi_1$ such that $\|\tilde{\Phi}_1 - \Phi_1\|_F$ is minimal
  4. Update $p \leftarrow 2 \times p$
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  6. Prune $\tilde{\Phi}_2$ to obtain $\Phi_2$ such that $\|\tilde{\Phi}_2 - \Phi_2\|_F$ is minimal
   
  - Get a set of scaling functions $\Phi_j$ spanning spaces $V_j$. 
Critically sampled wavelets

- Use the diffusion operator $T = \prod^{1/2} P \prod^{-1/2}$ to find the bases (scaling functions) $\{\Phi_j\}_{1 \leq j \leq J}$ of nested spaces $\{V_j\}_{1 \leq j \leq J}$, $V_J \subset V_{J-1} \cdots \subset V_0$.

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  : 

- Get a set of scaling functions $\Phi_j$ spanning spaces $V_j$.

- Diffusion wavelets $\Psi_j$ are obtained as the bases of the complement $W_j$ of $V_{j+1}$ in $V_j$

  $$\Psi_j = \Phi_j - \Phi_{j+1} \Phi_j^t \Phi_{j+1}$$
Some wavelets and scaling functions with diffusion method on cycle graph

**Figure 10.** Orthogonal and biorthogonal scaling functions on the directed cycle graph $C_{256}$. 
Some wavelets and scaling functions with diffusion method on cycle graph

**Figure 11.** Orthogonal and biorthogonal wavelet functions on the directed cycle graph $C_{256}$. 
Comparison between scaling functions from Graph Filters and from diffusion

- **Case 2:** Use the spectral properties of $\bar{T}_\alpha = \Pi^{1/2} \bar{P}_\alpha \Pi^{-1/2}$ to define:
  
  $$H_\alpha = \sum_{\omega \in \omega} h(tw) S_{w,\alpha}$$

  With $t = 2^4$ and $h(x) = \exp(-x)$.

- **Case 3:** For different scales $j$: $\{ T^{2^j} \}_{j=1}^5$ for *diffusion wavelet* and $\{ \bar{T}^{2^j} \}_{j=1}^5$ for *spectral wavelets*
Comparison between scaling functions from Graph Filters and diffusion.
Comparison between scaling functions from Graph Filters and from diffusion

**Figure 14.** Orthogonal and bi-orthogonal scaling functions built w.r.t the diffusion wavelet framework versus scaling function built w.r.t spectral wavelets framework.
Semi-supervised Learning

- **Method 1**: for $y$ in $\ell^2(\mathcal{V})$

\[
\arg\min_f c\|M_l(f - y)\|^2 + c\|(I - M_l)f\|^2 + \rho_2 \langle f, \mathcal{L}f \rangle
\]

Where $M_l$ is the diagonal matrix with 0 on vertices with unknown labels

- **Method 2**: for $y$ in $\ell^2(\mathcal{V}, \pi)$

\[
\arg\min_f c\|M_l(f - y)\|_\pi^2 + c\|(I - M_l)f\|_\pi^2 + \rho_2 \langle f, \mathcal{L}_{RW}f \rangle_{\pi}
\]

- **Method 3**: baseline method from [SM13]

\[
\arg\min_f c\|M_l(f - y)\|^2 + c\|f - W^{\text{norm}}f\|^2
\]

- **Benchmark**: graph of political blogs with binary label $-1/1$
Semi-supervised Learning
Graph Signal Reconstruction

- Random graph signal $y$ with missing values. Objective:

$$\arg\min_{\theta = \{\theta_k\}} \mathbb{E} \left[ \| f_0 - \sum_k \theta_k R^k y \|^2 \right]$$

- Results with different reference operator $R \in \{ P, \bar{P}, T, \bar{T}, W^{\text{norm}} \}$
Graph Signal Reconstruction
Conclusion

- The Wandom Walk Operator $P$ leads to a definition of the Laplacian $\mathcal{L}_{RW}$.

- $\mathcal{L}_{RW}$ is linked with the Dirichlet energy on $G$ which gives a definition of frequencies when applied on the eigenvectors of $P$.

- Graph Filters can be defined by constructing polynomial sum of $P$ or by linear combination of its eigenprojectors.

- Diffusion Wavelets can be constructed via the diffusion operator $T$, the proposed construction framework has some limitation.

- Good results on graph signal reconstruction and semi-supervised learning.

- The ergodic constraint on $P$ might not be necessary, maybe irreducible is sufficient.
Conclusion

- The Wandom Walk Operator $P$ leads to a definition of the Laplacian $\mathcal{L}_{RW}$.
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