Wavelets and Applications

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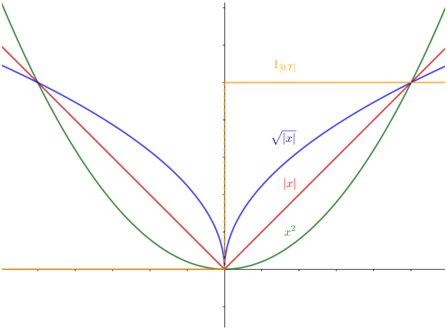
Wavelet zoom
a local characterization of functions

Local characterization of regularity via the derivatives

"Smoothness" depends on the differentiability class to which a function belongs to. Among these 4 continuous ($\mathcal{C}^0$) functions:

- $x \mapsto x^2$ is the only one differentiable everywhere and $\mathcal{C}^\infty$
- $x \mapsto |x|$ is not differentiable at $x = 0$ (corner)
- $x \mapsto \sqrt{|x|}$ (cusp) and $\mathbb{1}_{[0,T]}$ (jump) have kind of "infinite gradient" at the singularity point $x = 0$



Prerequisite: Global regularity through Fourier coefficients

Lemma (Riemann-Lebesgue)

If f is L^1 then the Fourier transform of f satisfies

$$\widehat{f}(\omega) = \int f(x) \mathrm{e}^{-i\omega x} \xrightarrow[|\omega| \to \infty]{} 0$$

How fast the Fourier coefficients decrease?

For f p times continuously differentiable with bounded derivatives, since $\widehat{f}(\omega) = \frac{1}{i\omega} \widehat{\frac{\mathrm{d}}{\mathrm{d}x} f}(\omega)$ then by iterating we get $\widehat{f}(\omega) = \frac{1}{(i\omega)^p} \widehat{\frac{\mathrm{d}^p}{\mathrm{d}x^p} f}(\omega)$

$$|\widehat{f}(\omega)| \leqslant \frac{K}{|\omega|^p}$$

with $K = \sup \frac{\mathrm{d}^p}{\mathrm{d}x^p} f$

Prerequisite: Global regularity through Fourier coefficients

Conversely Fourier decay governs smoothness?

If $\widehat{f}$ is L^1 then $f \in L^\infty$ and f is continuous.

Proof:

$$|f(x)| \leqslant \frac{1}{2\pi} \int |\mathrm{e}^{i\omega x} \widehat{f}(\omega)| \, \mathrm{d}\omega \leqslant \frac{1}{2\pi} \int |\widehat{f}(\omega)| \, \mathrm{d}\omega < +\infty$$

which proves boundedness. As for continuity, consider a sequence $y_n \to 0$ and

$$f(x - y_n) = \frac{1}{2\pi} \int \mathrm{e}^{i\omega(x - y_n)} \widehat{f}(\omega) \, \mathrm{d}\omega$$

The integrand converges pointwise to $\mathrm{e}^{i\omega x} \widehat{f}(\omega)$ and is uniformly bounded in modulus by the integrable function $\widehat{f}$. Hence Lebesgue's dominated convergence theorem applies and yields $f(x - y_n) \to f(x)$ that is continuity in x. $\square$

Prerequisite: Global regularity through Fourier coefficients

Theorem (Sufficiant condition for differentiability of f at order p)

A function f is bounded and p times continuously differentiable with bounded derivatives if

$$\int_{-\infty}^{\infty} |\widehat{f}(\omega)|(1+|\omega|^p)\,\mathrm{d}\omega < +\infty$$

Proof: Knowing that $\widehat{f^{(k)}} : \omega \mapsto (i\omega)^k \widehat{f}(\omega)$, by the inversion formula

$$|f^{(k)}(t)| = \left| \int_{-\infty}^{\infty} \widehat{f^{(k)}}(\omega) \mathrm{e}^{i\omega t}\,\mathrm{d}\omega \right| \leqslant \int_{-\infty}^{\infty} |\widehat{f}(\omega)| \cdot |\omega|^k\,\mathrm{d}\omega < +\infty$$

for any $k \leqslant p$, so $f^{(k)}$ is continuous and bounded. $\square$

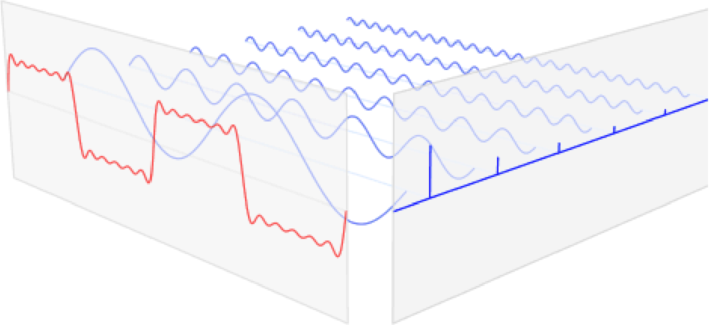
Corrolary. If it exists a constant K and $\epsilon > 0$ such that

$$|\widehat{f}(\omega)| \leqslant \frac{K}{1+|\omega|^{p+1+\epsilon}}, \quad \text{then} \quad f \in \mathcal{C}^p$$

Credits: S. Mallat (Wavelet tour)

Prerequisite: Global regularity through Fourier coefficients

The decay of $|\widehat{f}(\omega)|$ depends on the worst singular behavior of f



$$f(x) = \begin{cases} -1 \text{ if } -\pi \leqslant x < 0 \\ +1 \text{ if } 0 \leqslant x < \pi \end{cases} = \sum_{n=1}^{+\infty} \frac{4}{\pi(2n-1)} \sin((2n-1)x)$$

where f is periodized. For $f = \mathbb{1}_{[-T,T]} \Rightarrow |\widehat{f}(\omega)| = o(|\omega|^{-1})$

Credits: Wikipedia (https://en.wikipedia.org/wiki/Fourier_series)

Wavelet zoom: Lipschitz regularity

Definition (Lipschitz regularity of order α of a function f)

Let $\alpha \geq 0$ be the regularity parameter and $x_0 \in \mathbb{R}$.
f is pointwise Lipschitz–α at x_0, if there exist $C > 0$ and a polynomial P_n of degree $n = \lfloor \alpha \rfloor$, such that

$$\forall h \in \mathbb{R}, \quad |f(x_0 + h) - P_n(h)| \leq C|h|^\alpha \tag{1}$$

P_n is the Taylor expansion of f at x_0. (If $0 < \alpha < 1$, $P_0(h) = f(x_0)$)

- f is uniformly Lipschitz–α over $[a, b]$ if f satisfies (1) for all $x_0 \in [a, b]$, with a constant C independent of x_0.
- Extension to negative α (distributions): f uniformly Lipschitz–α over $]a, b[$ if its primitive is Lipschitz–$(\alpha + 1)$ over $]a, b[$.
- The Lipschitz regularity of f is the supremum of the α such that f is Lipschitz–α.

Lipschitz–α functions

$$\forall h \in \mathbb{R}, \quad |f(x_0 + h) - f(x0)| \leq C|h|^{\alpha}$$

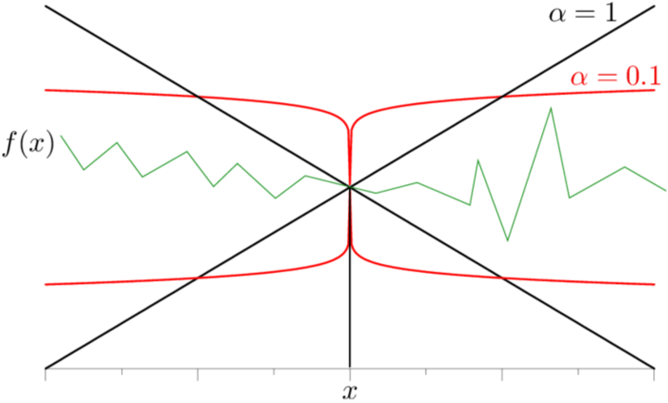
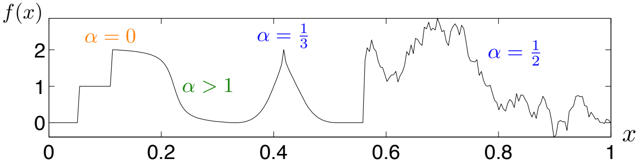


Figure: The schematic diagram of Lipschitz–α functions

Credits: Li-Wei Liu & Hong-Ki Hong

Some examples

- A Lipschitz–α function at x_0, with $0 < \alpha < 1$, is continuous, but a priori non differentiable.
- A $\mathcal{C}^1$ function in a neighborhood of x_0 is Lipschitz–1 at x_0.
- The Lipschitz regularity α with $n < \alpha < n+1$ allows to classify regularities between $\mathcal{C}^n$ and $\mathcal{C}^{n+1}$.
- A bounded function is Lipschitz–0. For example the Heavyside function $H(x) = 1$ if $x \geq 0$ and 0 if $x < 0$.
- The distribution δ is Lipschitz–(-1) (as the derivative of H).
- The function $x \mapsto |x - x_0|^\alpha$ $(0 < \alpha < 1)$ is Lipschitz–α
- The function $\sqrt{|\cos(2\pi x)|}$ is Lipschitz–$\frac{1}{2}$.



Some examples

A Holder function of exponant $\frac{1}{2}$

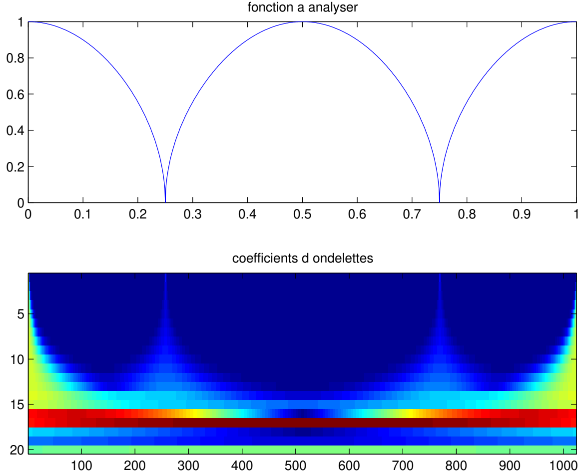


Figure: $f(x) = \sqrt{|\cos(2\pi x)|}$ and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)

Some examples

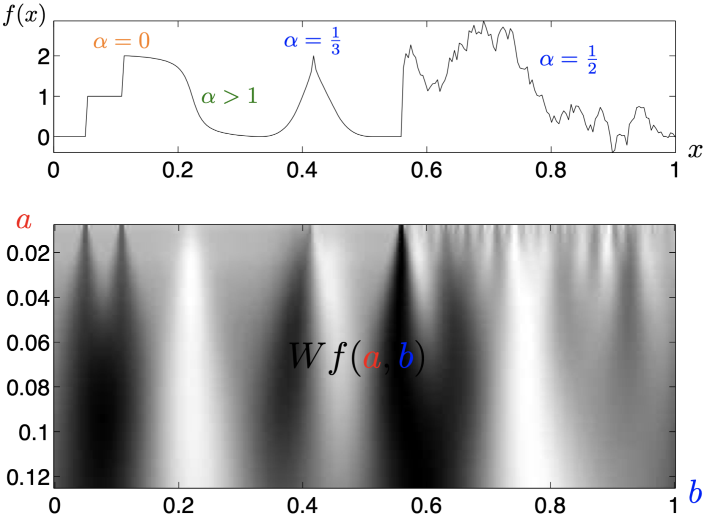


Figure: Wavelet transform $Wf(a, b)$ calculated with $\psi = -\theta'$ where θ is a Gaussian. Singularities create large amplitude coefficients in their cone influence.

Credits: S. Mallat (Wavelet tour)

Regularity measurements with wavelets

Let $\alpha \geq 0$ be fixed, ψ a wavelet with compact support $\subset [-L, L]$, and $N > \alpha$ vanishing moments:

$$\int x^n \psi(x)\, \mathrm{d}x = 0, \quad \text{for } 0 \leqslant n < N$$

Remark: a wavelet with N vanishing moments is orthogonal to polynomials of degree $N-1$.

Polynomial Suppression. Let f Lipschitz-α at x_0, that is

$$f(x) = P_n(x - x_0) + \varepsilon(x - x_0) \quad \text{with} \quad |\varepsilon(x - x_0)| \leq |x - x_0|^\alpha$$

Since $\alpha < N$, the polynomial P_N has degree at most $N-1$.
With the change of variable $y = (x-b)/a$, we verify that

$$WP_n(a, b) = \int_{-\infty}^{+\infty} P_n(x) \frac{1}{\sqrt{a}} \psi\left(\frac{x-b}{a}\right) \mathrm{d}x = 0$$

Then,

$$Wf(a, b) = W\varepsilon(a, b)$$

Pointwise Lipschitz regularity and wavelet coefficients

Let $\alpha \geq 0$. One consider a wavelet ψ of regularity $\mathcal{C}^N$, with compact support $\operatorname{supp}\, \psi \subset [-L, L]$, and $N \geq \alpha$ vanishing moments.

Theorem (Jaffard, Estimation of the local regularity of f at point x_0)

If $f \in L^2(\mathbb{R})$ is Lipschitz–$\alpha \leq N$ at x_0, then $\exists A > 0$ such that

$$\forall (a, b) \in \mathbb{R} \times \mathbb{R}^+, \quad |Wf(a, b)| \leqslant A\ a^{\alpha + \frac{1}{2}} \left(1 + \left| \frac{b - x_0}{a} \right|^{\alpha} \right)$$

Conversely, if $\alpha < N$ is not an integer and there exist $A > 0$ and $\alpha' < \alpha$ such that

$$\forall (a, b) \in \mathbb{R} \times \mathbb{R}^+, \quad |Wf(a, b)| \leqslant A\ a^{\alpha + \frac{1}{2}} \left(1 + \left| \frac{b - x_0}{a} \right|^{\alpha'} \right)$$

then f is Lipschitz-α at x_0.

Proof of $\boxed{\Rightarrow}$

Since f is Lipschitz–α at x_0, there exists a polynomial P_N of degree $\lfloor \alpha \rfloor < N$ and $C > 0$ such that

$$|f(x) - P_N(x - x_0)| \leqslant C|x - x_0|^\alpha$$

Since ψ has N vanishing moments, we saw that $WP_n(a, b) = 0$, and thus

$$\begin{aligned} |Wf(a, b)| &= \left| \int_{-\infty}^{\infty} [f(x) - P_N(x - x_0)]\, \psi_{a,b}(x)\, \mathrm{d}x \right| \\ &\leqslant \int C|x - x_0|^\alpha \frac{1}{\sqrt{a}} \left| \psi\left(\frac{x - b}{a}\right) \right| \mathrm{d}x \end{aligned}$$

The change of variable $y = \frac{x-b}{a}$ gives

$$|Wf(a, b)| \leqslant \sqrt{a} \int_{-\infty}^{\infty} C|ay + b - x_0|^\alpha |\psi(y)|\, \mathrm{d}y$$

Proof of $\boxed{\Rightarrow}$

$$|Wf(a,b)| \leqslant \sqrt{a}\int_{-\infty}^{\infty} C|\underbrace{ay}_{t} + \underbrace{b - x_0}_{s}|^{\alpha}|\psi(y)|\,\mathrm{d}y$$

Lemma: $|t+s|^{\alpha} \leqslant 2^{\alpha}(|t|^{\alpha} + |s|^{\alpha})$

Proof: Let $m = \max(|t|,|s|)$ so that $|t+s| \leqslant |t| + |s| \leqslant 2m$. Then,

$$|t+s|^{\alpha} \leqslant (2m)^{\alpha} = 2^{\alpha}m^{\alpha} \leqslant 2^{\alpha}(|t|^{\alpha} + |s|^{\alpha})\ .$$

By the lemma,

$$\begin{aligned}
|Wf(a,b)| &\leqslant C2^{\alpha}\sqrt{a}\left(a^{\alpha}\int_{-\infty}^{\infty}|y|^{\alpha}|\psi(y)|\,\mathrm{d}y + |b-x_0|^{\alpha}\int_{-\infty}^{\infty}|\psi(y)|\,\mathrm{d}y\right)\\
&\leqslant \underbrace{KM2^{\alpha}}_{A}\, s^{\alpha+\frac{1}{2}}\left(1 + \left|\frac{b-x_0}{a}\right|^{\alpha}\right)
\end{aligned}$$

with $M = \max\left(\int_{-\infty}^{\infty}|y|^{\alpha}|\psi(y)|,\ \int_{-\infty}^{\infty}|\psi(y)|\,\mathrm{d}y\right)$. $\square$

Cone of Influence

If $\mathrm{supp}\ \psi = [-L, L]$, the cone of influence of x_0 in the time-scale space is the set of points such that $x_0 \in \mathrm{supp}\ \psi_{a,b} = [b - La, b + La]$, that is

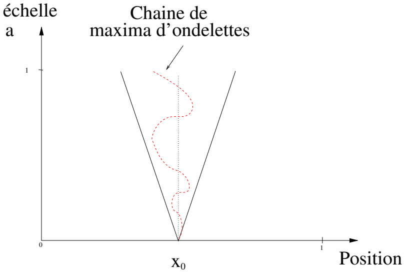
$$\Gamma(x_0) = \{(b, a) \in \mathbb{R} \times \mathbb{R}_+^* \ : \ |b - x_0| < La\}$$

If f is Lipschitz-α at x_0, then $\exists A > 0$, such that for all $(b, a) \in \Gamma(x_0)$:

$$|Wf(a, b)| \leq A\ a^{\alpha + \frac{1}{2}}$$

and conversely for α non integer.

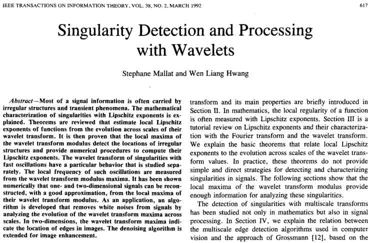
α is computed by the slope of the curve $\log a \to \log |Wf(a, b)|$



Wavelet Transform Modulus Maxima

References

- S. Mallat, W.L. Hwang *Singularity detection and processing with wavelet*, IEEE Trans. Info. Theory, 38(2):617-643, Mars 1992
- S. Mallat, S.Zhong *Characterization of Signals from Multiscale Edges*, IEEE Trans. Patt. Anal. and Mach. Intell., 14(7):710-732, Juillet 1992



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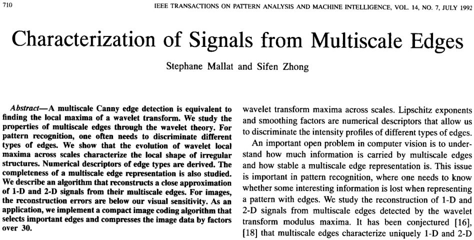
Singularity Detection and Processing with Wavelets

Stephane Mallat and Wen Liang Hwang

Abstract—Most of a signal information is often carried by irregular structures and transient phenomena. The mathematical characterization of singularities with Lipschitz exponents is explained. Theorems are reviewed that estimate local Lipschitz exponents of functions from the evolution across scales of their wavelet transform. It is then proven that the local maxima of the wavelet transform modulus detect the locations of irregular structures and provide numerical procedures to compute their Lipschitz exponents. The wavelet transform of singularities with fast oscillations have a particular behavior that is studied separately. The local frequency of such oscillations are measured from the wavelet transform modulus maxima. It has been shown numerically that one- and two-dimensional signals can be reconstructed, with a good approximation, from the local maxima of their wavelet transform modulus. As an application, an algorithm is developed that removes white noises from signals by analyzing the evolution of the wavelet transform maxima across scales. In two-dimensions, the wavelet transform maxima indicate the location of edges in images. The denoising algorithm is extended for image enhancement.

transform and its main properties are briefly introduced in Section II. In mathematics, the local regularity of a function is often measured with Lipschitz exponents. Section III is a tutorial review on Lipschitz exponents and their characterization with the Fourier transform and the wavelet transform. We explain the basic theorems that relate local Lipschitz exponents to the evolution across scales of the wavelet transform values. In practice, these theorems do not provide simple and direct strategies for detecting and characterizing singularities in signals. The following sections show that the local maxima of the wavelet transform modulus provide enough information for analyzing these singularities.

The detection of singularities with multiscale transforms has been studied not only in mathematics but also in signal processing. In Section IV, we explain the relation between the multiscale edge detection algorithms used in computer vision and the approach of Grossmann [12], based on the



710 IEEE TRANSACTIONS ON PATTERN ANALYSIS AND MACHINE INTELLIGENCE, VOL. 14, NO. 7, JULY 1992

Characterization of Signals from Multiscale Edges

Stephane Mallat and Sifen Zhong

Abstract—A multiscale Canny edge detection is equivalent to finding the local maxima of a wavelet transform. We study the properties of multiscale edges through the wavelet theory. For pattern recognition, one often needs to discriminate different types of edges. We show that the evolution of wavelet local maxima across scales characterize the local shape of irregular structures. Numerical descriptors of edge types are derived. The completeness of a multiscale edge representation is also studied. We describe an algorithm that reconstructs a close approximation of 1-D and 2-D signals from their multiscale edges. For images, the reconstruction errors are below our visual sensitivity. As an application, we implement a compact image coding algorithm that selects important edges and compresses the image data by factors over 30.

wavelet transform maxima across scales. Lipschitz exponents and smoothing factors are numerical descriptors that allow us to discriminate the intensity profiles of different types of edges.

An important open problem in computer vision is to understand how much information is carried by multiscale edges and how stable a multiscale edge representation is. This issue is important in pattern recognition, where one needs to know whether some interesting information is lost when representing a pattern with edges. We study the reconstruction of 1-D and 2-D signals from multiscale edges detected by the wavelet transform modulus maxima. It has been conjectured [16], [18] that multiscale edges characterize uniquely 1-D and 2-D

Wavelet construction from the derivatives of a Gaussian

Let $\theta(x) = \exp(-x^2/\sigma^2)$ the Gaussian Kernel and let considered

$$\psi^N(x) \equiv \theta^{(n)}(x) = \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^N \mathrm{e}^{-\frac{x^2}{\sigma^2}}$$

The wavelet ψ^N has N vanishing moments.

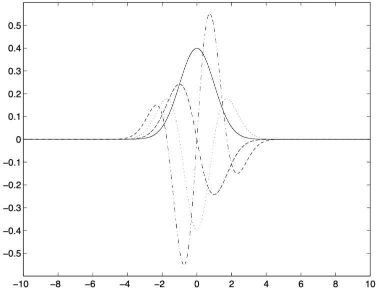


Figure: The Gaussian θ ($n = 0$) for $\sigma = 1$ and its two first derivatives: $n = 1$ is represented in $(- \cdot -)$ and $n = 2$ (the Mexican hat) in $(\cdots)$

Multiscale differential operator

A wavelet ψ has fast decay if

$$\forall m \in \mathbb{N}, \quad \exists C_m \quad \text{such that} \quad |\psi(x)| \leqslant \frac{C_m}{1+|t|^m}, \quad \forall t \in \mathbb{R}$$

Theorem (Multiscale differential operator)

A wavelet ψ with fast decay has N vanishing moments if and only if there exists θ with a fast decay such that

$$\psi(x) = (-1)^N \frac{\mathrm{d}^N \theta}{\mathrm{d}x^N}(x)$$

As a consequence

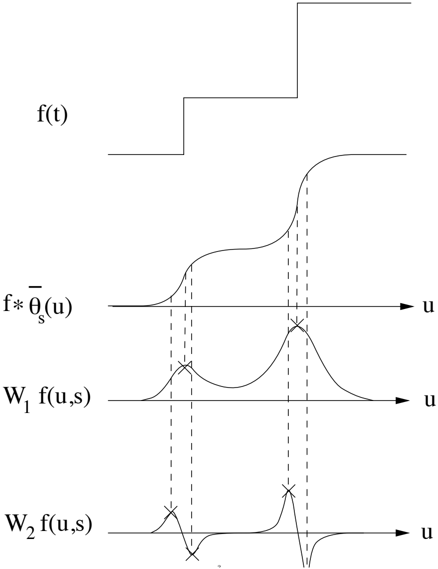
$$W_N f(a,b) = a^N \frac{\mathrm{d}^N \theta}{\mathrm{d}b^N}(f * \check{\theta}_a)(b)$$

Moreover, ψ has no more vanishing moments iff $\int \psi \neq 0$.

Multiscale differential operator

Example

- The convolution $f * \check{\theta}_a$ averages f over a domain proportional to a
- If the wavelet has only one vanishing moment: $\psi = -\theta'$ then $W_1(a, b) = a\frac{\mathrm{d}}{\mathrm{d}b}(f * \check{\theta}_a)(b)$ has modulus maxima at sharp variation points of $f * \check{\theta}_a$
- If the wavelet has two vanishing moment: $\psi = -\theta''$ then $W_2(a, b) = a\frac{\mathrm{d}^2}{\mathrm{d}b^2}(f * \check{\theta}_a)(b)$ corresponds to locally maximum curvatures



Wavelet Maxima Lines

- Point of Modulus Maximum are any point (b_0, a_0) in the time-scale plan such that the curve $b \mapsto |Wf(b, a_0)|$ is locally maximum at $b = b_0$. This implies that

$$\frac{\partial Wf(a_0, b_0)}{\partial b} = 0$$

- Maxima lines is any connected curve $a(b)$ in the scale-space plane (b, a) along which all points are modulus maxima.

Theorem (Hwang, Mallat)

Suppose that ψ is $\mathcal{C}^N$ with a compact support and $\psi = (-1)^N \theta^{(N)}$ with $\int \theta \neq 0$. Let $f \in L^1[b_0, b_1]$. If there exists $a_0 > 0$ such that $|Wf(a, b)|$ has no local maximum for $b \in [b_0, b_1]$ and $a < a_0$, then f is uniformly Lipschitz–N on $[b_0 + \epsilon, b_1 - \epsilon]$, for any $\epsilon > 0$.

Wavelet Maxima Lines

Remarks

- This theorem implies that f can be singular (not Lipschitz–1) at a point x_0 only if there is a sequence of wavelet maxima points $(b_k, a_k)_{k\in\mathbb{N}}$ that converges toward x_0 at fine scales:

$$\lim_{k\to+\infty} b_k = \mathbf{x}_0 \quad \text{and} \quad \lim_{k\to+\infty} a_k = 0$$

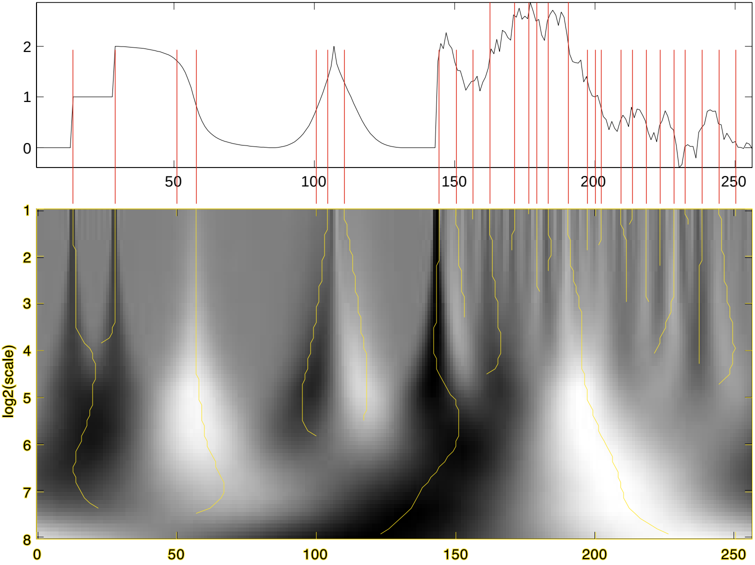
- These modulus maxima points may or may not be along the same maxima line. This result guarantees that all singularities are detected by following the wavelet transform modulus maxima at fine scales

Theorem (Hummel, Poggio, Yuille)

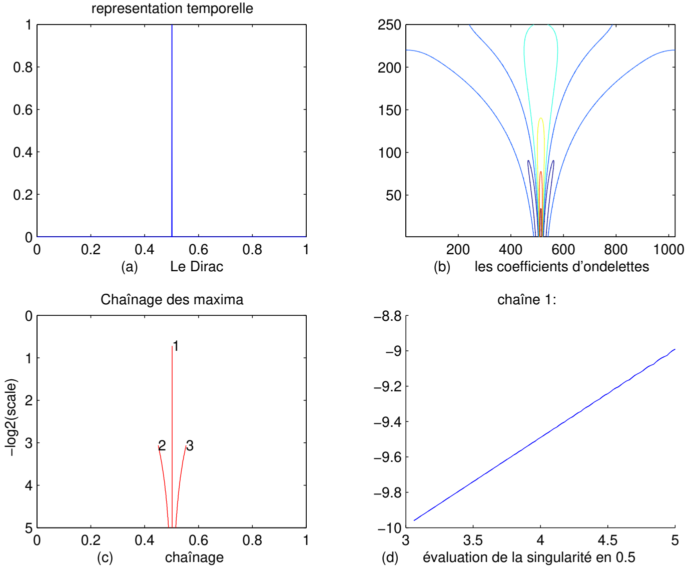
Let $\psi = (-1)^N \theta^{(N)}$ where θ is Gaussian. For any $f \in L^2$, the modulus maxima of $Wf(a,b)$ belongs to connected curves that are never interrupted when the scale decreases

Wavelet Maxima Lines

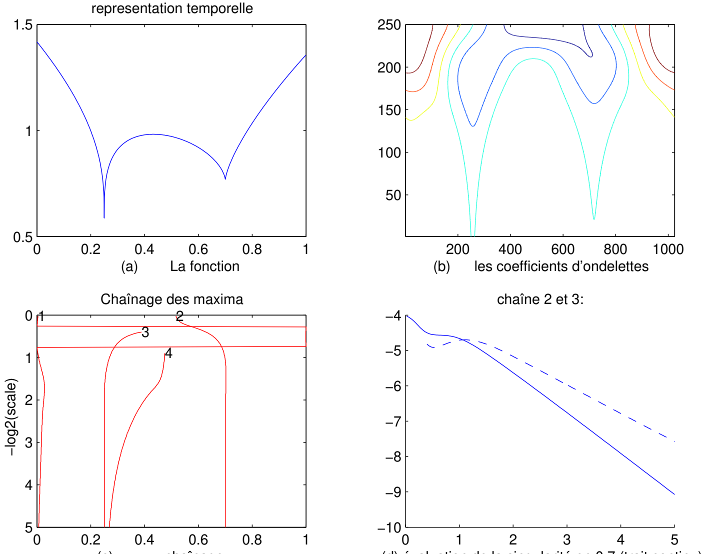
Example



Example: a simple Dirac δ



Example: 2 cusps $f(x) = |x - 0.25|^{\frac{1}{3}} + |x - 0.7|^{\frac{2}{3}}$



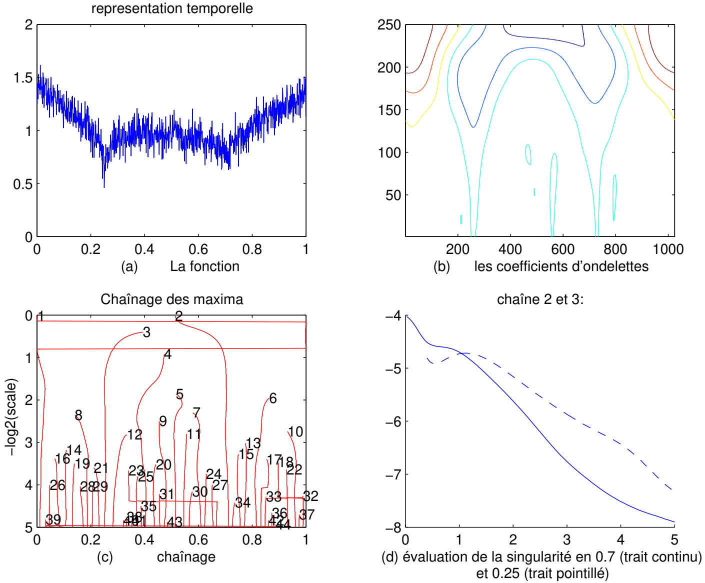
(a) La fonction

(b) les coefficients d'ondelettes

(c) chaînage

(d) évaluation de la singularité en 0.7 (trait continu) et 0.25 (trait pointillé)

Example: $f(x) = |x - 0.25|^{\frac{1}{3}} + |x - 0.7|^{\frac{2}{3}}$+noise (SNR=0.01)



Practical estimation of α

f is uniformly Lipschitz–α in the neighborhood of x_0 iff there exists $A > 0$ such that each modulus maximum (b, a) in the cone satisfies

$$|Wf(a, b)| \leqslant A\, a^{\alpha + \frac{1}{2}}$$

which is equivalent to

$$\log_2 |Wf(a, b)| \leqslant \log_2 A + \left(\alpha + \frac{1}{2}\right) \log_2 a$$

$\Rightarrow$ The Lipschitz regularity at x_0 is the maximum slope of $\log_2 |Wf(a, b)|$ as a function of $\log_2 a$ along the maxima lines converging to x_0

Practical estimation of α

Example

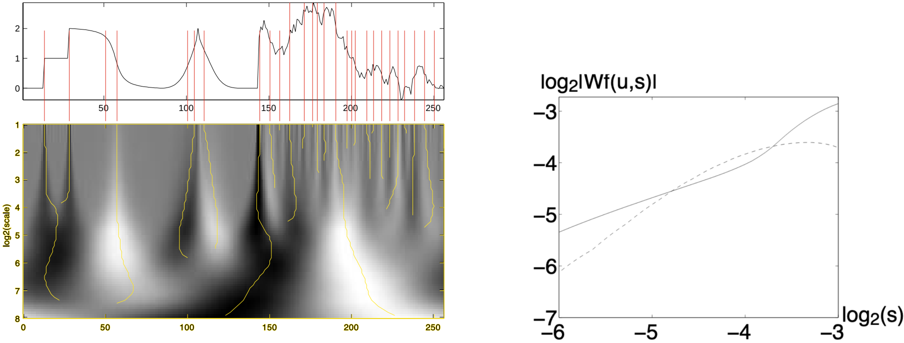
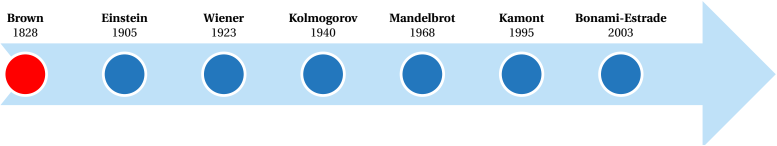
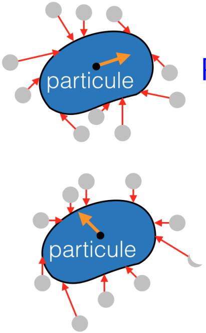


Figure: The full line gives the decay along the maxima line that converges to the first jump, and the dashed line to the first cusp.

Brownian motion

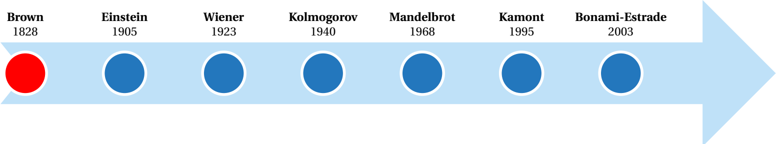




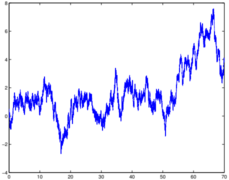
Properties

- Independants displacements
- Gaussian distribution
- Irregular trajectories

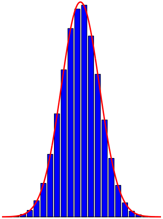
Brownian motion



Independants displacements

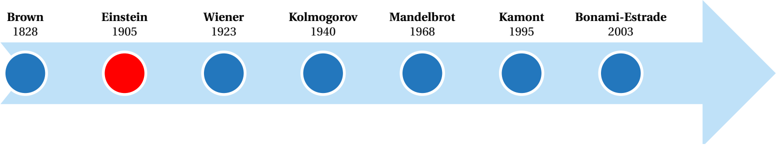


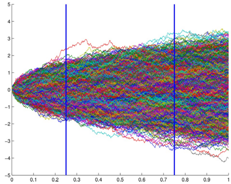
Irregular trajectories

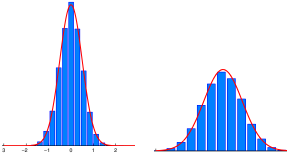


Gaussian distribution

Brownian motion



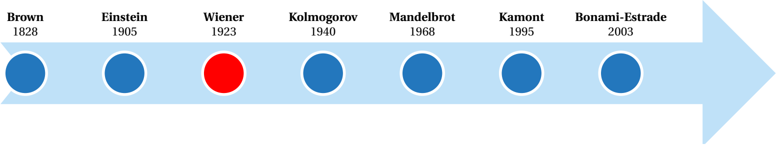




(a) (b)

$$\overline{(\Delta x)^2} \propto t$$

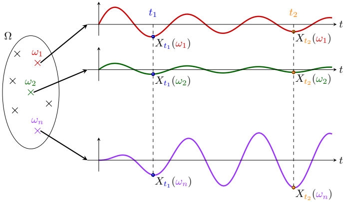
Brownian motion



$$\begin{array}{rcccl} X & : & T \times \Omega & \longrightarrow & E \\ & & (t,\omega) & \longmapsto & X(t,\omega) \end{array}$$

Brownian motion

- $(B_t)_t$ has independants increments, $B_0 = 0$ a.s.
- $B_{t_i} - B_{t_j} \sim \mathcal{N}(0, t_i - t_j)$
- $(B_t)_t$ has continuous sample paths a.s.



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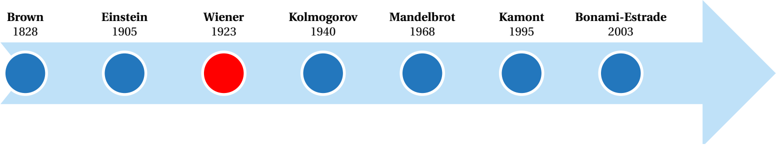
Brownian motion

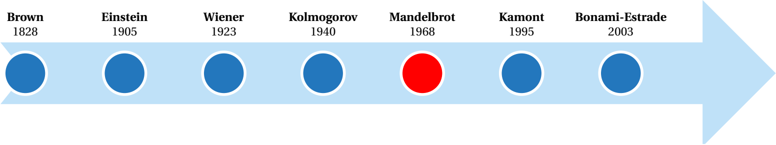


Isometry $\mathbf{W} : (L^2, \langle f, g \rangle_{L^2}) \to (\mathcal{G}, \mathbb{E}[XY])$

- $\mathbb{E}[\mathbf{W}(f)\mathbf{W}(g)] = \langle f, g \rangle_{L^2}, \quad \mathbf{W}(f) \sim \mathcal{N}(0, \|f\|_{L^2}^2)$
- $\forall t \in [0,1], \quad B_t \stackrel{\text{def}}{=} \mathbf{W}(\mathbb{1}_{[0,t]})$
- $\mathbb{E}\left[(B_t - B_s)^2\right] = \left\|\mathbb{1}_{[0,t]} - \mathbb{1}_{[0,s]}\right\|_{L^2}^2 = \int \mathbb{1}_{[s,t]} = t - s$
- $\mathbb{E}\left[(B_{t_i} - B_{t_i-1})(B_{t_j} - B_{t_j-1})\right] = \langle \mathbb{1}_{[t_{i-1},t_i]}, \mathbb{1}_{[t_{j-1},t_j]} \rangle_{L^2} = 0$

$$\text{Wiener stochastic integral} = \int f(x)\mathbf{W}(\mathrm{d}x)$$

Self-similarity

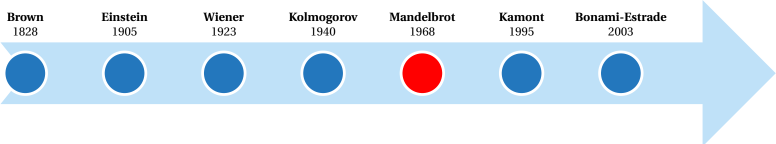


Self-similarity

$\{X(t)\}_{t \in T}$ self-similar of order H if

$$\forall \lambda \in \mathbb{R}, \ \{X(\lambda t)\}_{t \in T} \overset{(fdd)}{=} \lambda^H \{X(t)\}_{t \in T}$$

Self-similarity



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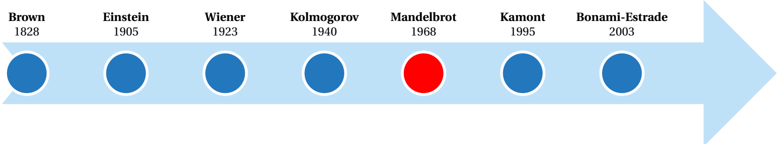

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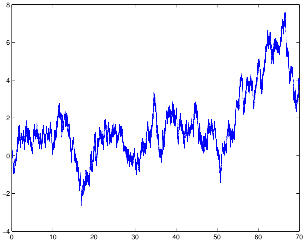
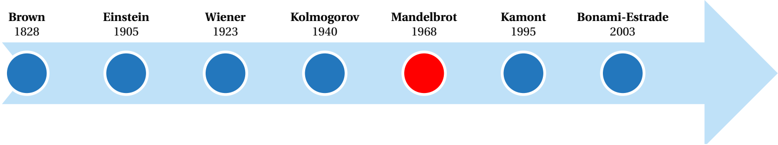
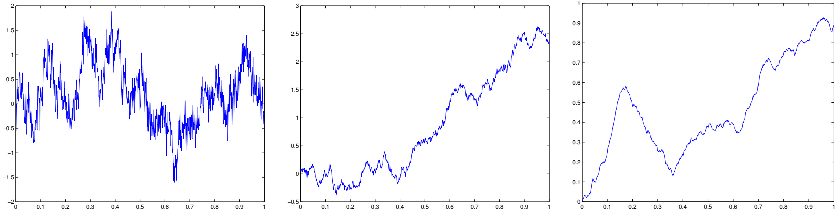


Fractional Brownian motion



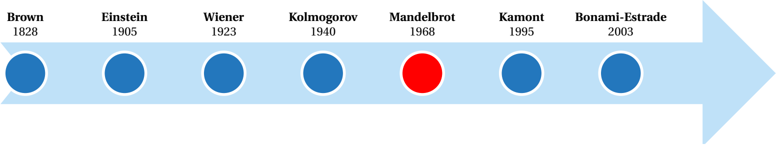
- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ ~~indpt. increments~~



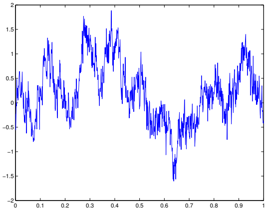
$H = 0.2$ $H = 0.5$ $H = 0.8$

Figure: Fractional Brownian motion B^H

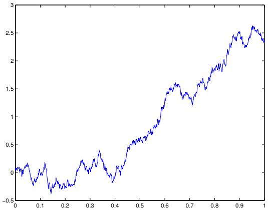
Fractional Brownian motion



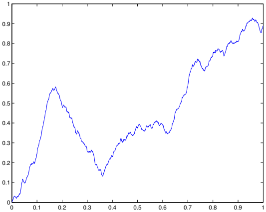
- $\mathbb{E}\left[\left(B^H(t) - B^H(s)\right)^2\right] = |t-s|^{2H} \Rightarrow$ stat. increments



$H = 0.2$

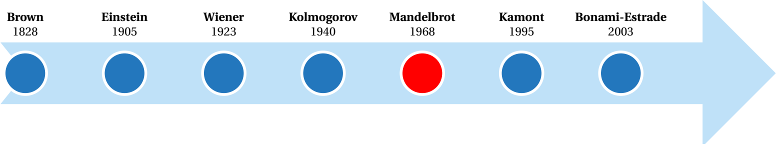


$H = 0.5$

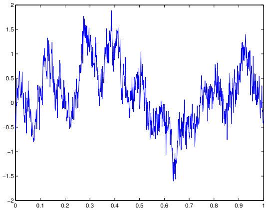


$H = 0.8$

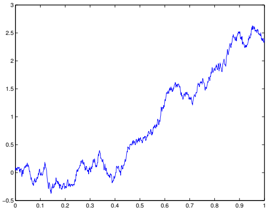
Fractional Brownian motion



- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ stat. increments
- $\mathbf{R}(t,s) = \mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$



$H = 0.2$



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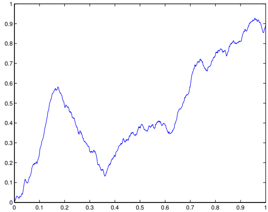
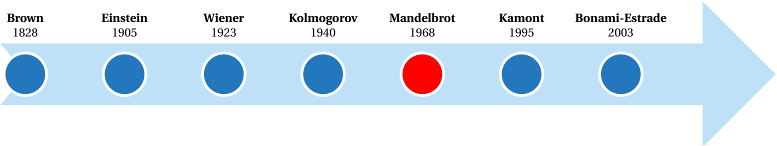
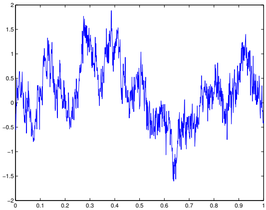


$H = 0.8$

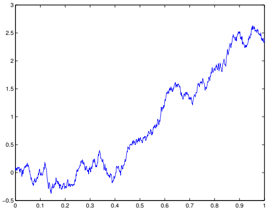
Fractional Brownian motion



- $\mathbb{E}\left[(B^H(t) - B^H(s))^2\right] = |t-s|^{2H} \Rightarrow$ stat. increments
- $\mathbf{R}(t,s) = \mathrm{Cov}(B^H(t), B^H(s)) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H})$
- $B^H(t) = \frac{1}{c_H} \int_{\mathbb{R}} \frac{\mathrm{e}^{\mathrm{j}t\xi}-1}{|\xi|^{H+1/2}} \widehat{\mathbf{W}}(\xi) \Rightarrow$ harmonizable formula



$H = 0.2$



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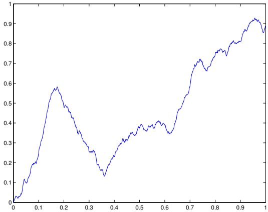
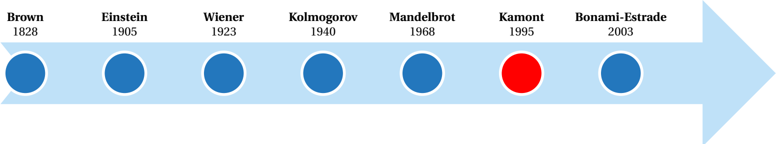
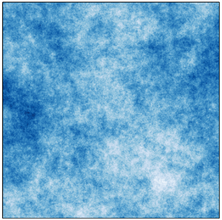


$H = 0.8$

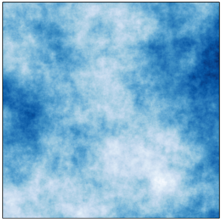
Fractional Brownian field



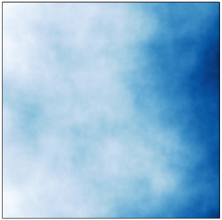
- $\mathbb{E}\left[(B^H(\boldsymbol{x}) - B^H(\boldsymbol{y}))^2\right] = \|\boldsymbol{x} - \boldsymbol{y}\|^{2H}$, $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$
- $\mathrm{R}(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\left(\|\boldsymbol{x}\|^{2H} + \|\boldsymbol{y}\|^{2H} - \|\boldsymbol{x} - \boldsymbol{y}\|^{2H}\right)$
- $B^H(\boldsymbol{x}) = \frac{1}{C_H} \int_{\mathbb{R}^2} \frac{\mathrm{e}^{\mathrm{j}\langle \boldsymbol{x}, \boldsymbol{\xi}\rangle} - 1}{\|\boldsymbol{\xi}\|^{H+1}} \widehat{\mathrm{W}}(\mathrm{d}\boldsymbol{\xi})$



$H = 0.2$



$H = 0.5$



$H = 0.8$

Wavelet-based estimation of the Hurst exponent

- Let us consider a discrete wavelet transform at scales $a = 2^{-j}$ and positions $b = k$

$$\psi_{j,k}(x) = 2^{-j/2}\psi(2^{-j}x - k)$$

which encodes series information in details

$$d_{j,k} = \langle B^H, \psi_{j,k} \rangle$$

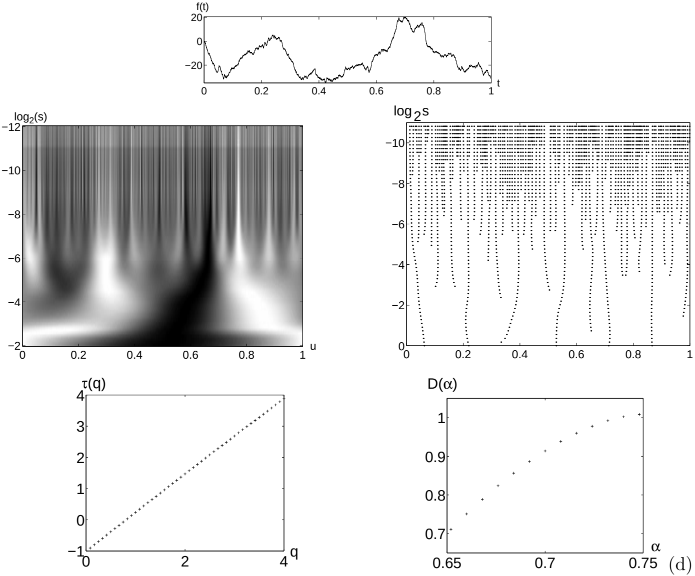
- Compute wavelet variance

$$\mathrm{Var}(d_{j,\bullet}) = \frac{1}{n_j} \sum_{k=0}^{n_j - 1} |d_{j,k}|^2$$

- Plot the $\log_2$ of variances versus scale j

$$\log_2(\mathrm{Var}(d_{j,\bullet})) = (2H+1)j + \text{cste}$$

Wavelet Maxima Lines for Brownian motion



Credits: S. Mallat (Wavelet tour)

Take home message

- Vanishing moments up to order N make the wavelet ψ blind to polynomial of degree $\leqslant N$ (smooth part of the signal), leading to better detections of singularities
- If the function is Lipschitz–α, then the amplitude of the wavelet coefficients are going to decay very fast to zero when the scale goes to zero (all the more that α is high)
- A remarkable aspect is the reverse: if we know this property, then we can characterize the pointwise regularity of the function at any point
- All singularities are detected by following the wavelet transform modulus maxima at fine scale
- The Lipschitz regularity at every point can be retrieved by measuring the maximum slop of the decay of $\log_2 |Wf(a, b)|$
- The wavelet-based estimation of the Lipschitz regularity enables to recover the self-similarity exponent of fractals