Wavelets and Applications

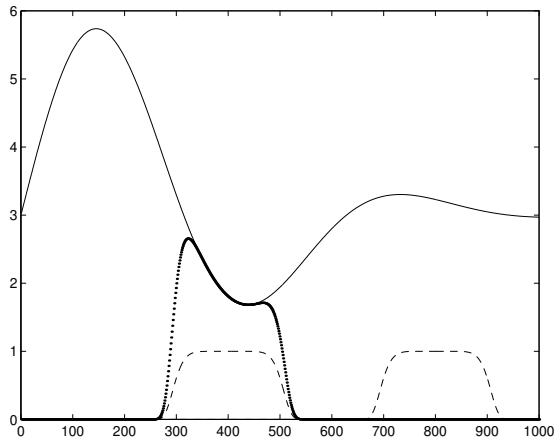
Kévin Polisano
kevin.polisano@univ-grenoble-alpes.fr

M2 MSIAM & Ensimag 3A MMIS

October 14, 2020

The Continuous Wavelet Transform

Short Time Fourier Transform (STFT)



Multiplication of the signal $f(x)$ by a window $w(x-b)$ (real and of size a_0) and computation of the Fourier transform of this product:

$$Sf(\nu, b) = \int_{-\infty}^{+\infty} f(x) w(x-b) \mathrm{e}^{-2i\pi\nu x} \, \mathrm{d}x$$

where b represents *time* and ν *frequency*. f can be recovered from its STFT coefficients:

$$f(x) = C_h \iint_{\mathbb{R}^2} Sf(\nu, b) w(x-b) \mathrm{e}^{2i\pi\nu x} \, \mathrm{d}\nu \, \mathrm{d}b$$

Special case: the Gabor Transform

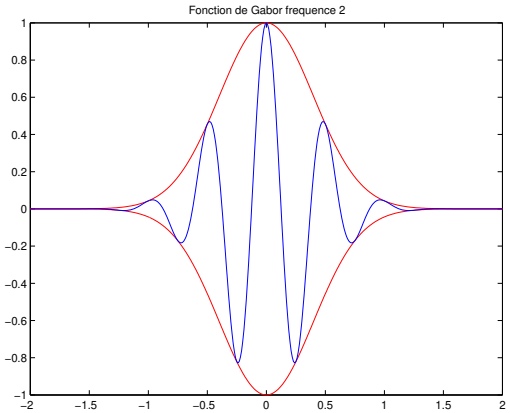
- In the Short Time Fourier Transform

$$Sf(\nu, b) = \int_{-\infty}^{+\infty} f(x) w(x-b) \mathrm{e}^{-2i\pi\nu x} \, \mathrm{d}x = \langle f, \psi_{\nu,b} \rangle$$

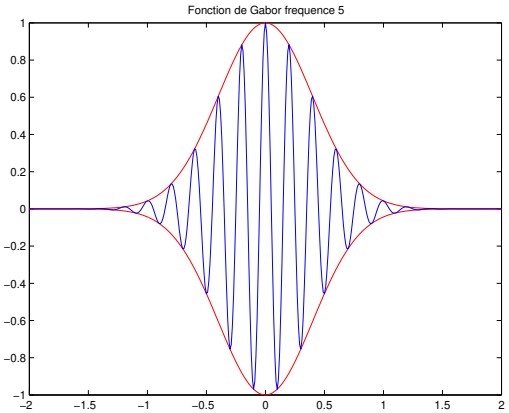
the analyzing functions are:

$$\psi_{\nu,b} = w(x-b)\mathrm{e}^{2i\pi\nu x}$$

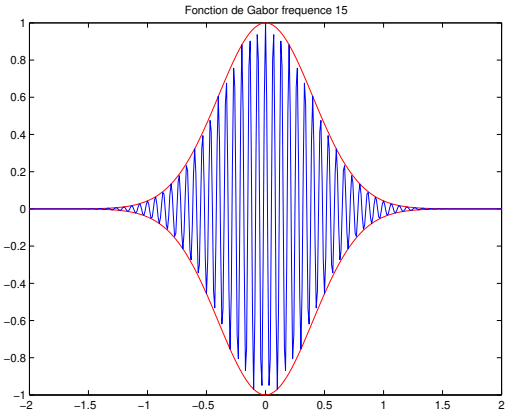
- In the **Gabor transform** (1946) the window w is a Gaussian of scale σ: $w(x) = \frac{1}{\sigma}\mathrm{e}^{-\pi(\frac{x}{\sigma})^2}$ and the Gabor functions are then ($\sigma = 1$):



(a) $\nu = 2$



(b) $\nu = 5$



(c) $\nu = 15$

Short Time Fourier Transform

Example of two musical notes

The time-frequency analysis allows to recover both frequencies (the notes) and temporal information (the temporal order) of the signal f_1:

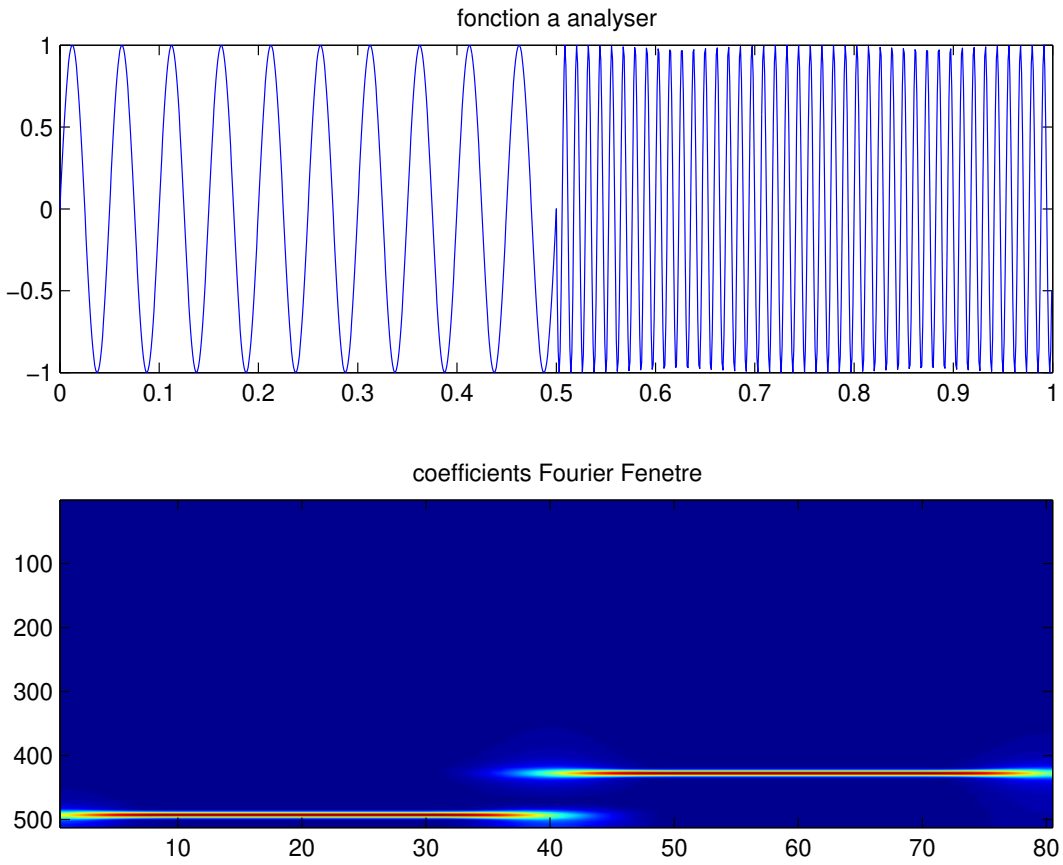
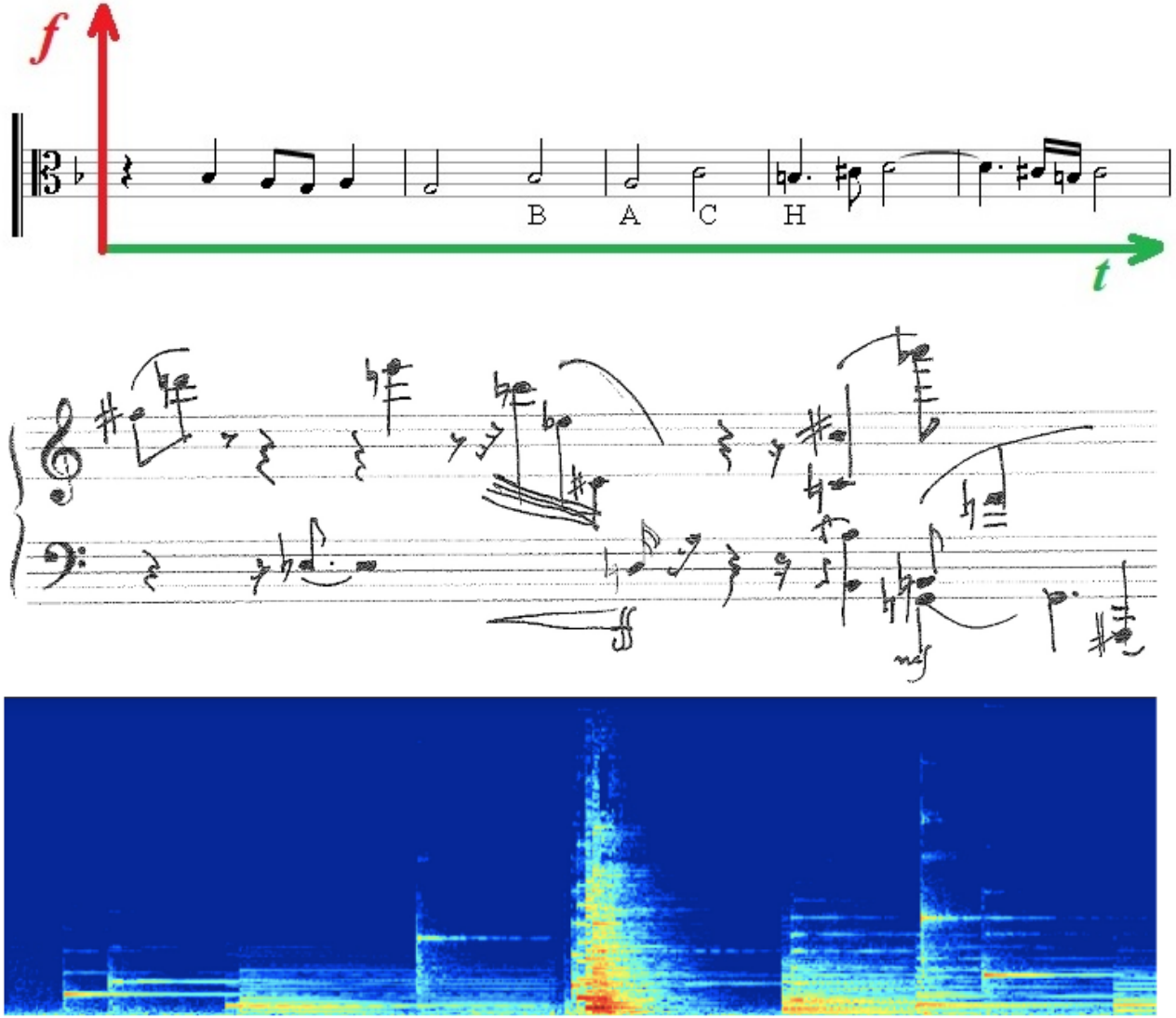


Figure: Time-frequency plane with b on the x-axis and ν on the y-axis, representing the density energy $|Sf(\nu, b)|^2 = |\langle f, g_{\nu,b}\rangle|^2$ called the spectrogram.

Short Time Fourier Transform

Analogy with music scores: an example with a piano



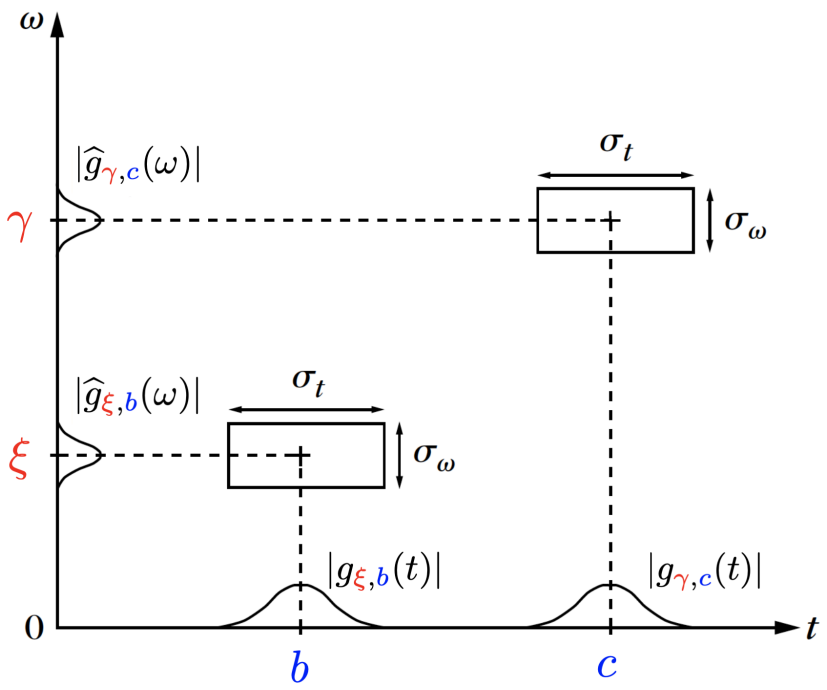
Credits: Patrick Flandrin, "Au-delà de Fourier, un monde qui vibre" (interstices.info)

Heisenberg boxes

Time-frequency localization and spread

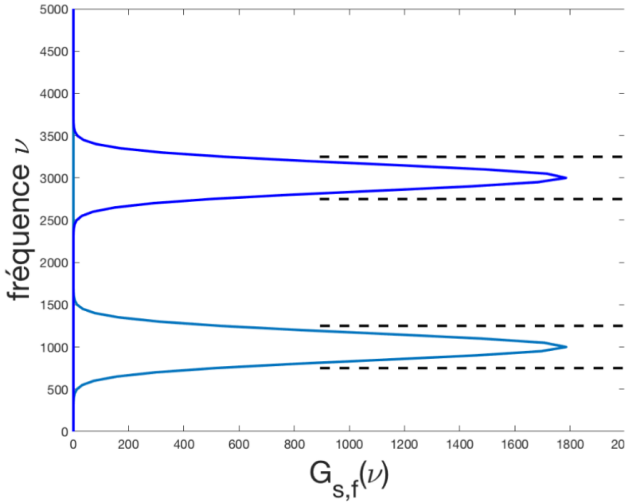
$$g_{\xi,b}(t) = w(t-b)\mathrm{e}^{i\xi t} \longleftrightarrow \widehat{g}_{\xi,b}(\omega) = \widehat{w}(\omega-\xi)\mathrm{e}^{-ib(\omega-\xi)}$$

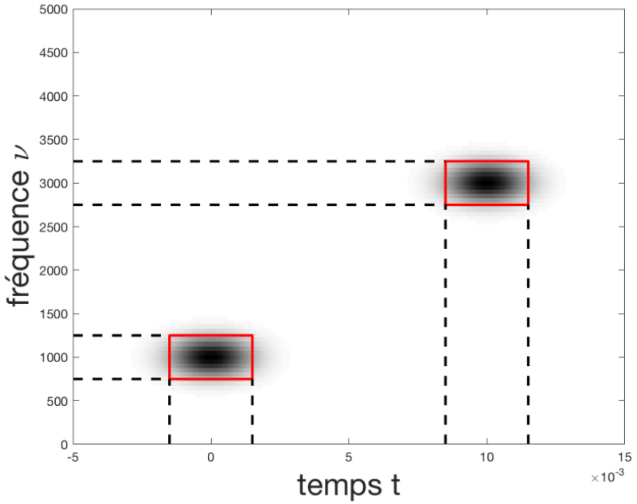
$$\begin{aligned}
\sigma_t^2 &= \int_{-\infty}^{\infty}(t-b)^2|g_{\xi,b}(t)|^2\,\mathrm{d}t = \int_{-\infty}^{\infty} t^2|w(t)|^2\,\mathrm{d}t \\
\sigma_\omega^2 &= \frac{1}{2\pi}\int_{-\infty}^{\infty}(\omega-\xi)^2|\widehat{g}_{\xi,b}(\omega)|^2\,\mathrm{d}\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2|\widehat{w}(t)|^2\,\mathrm{d}\omega
\end{aligned}$$



Heisenberg boxes

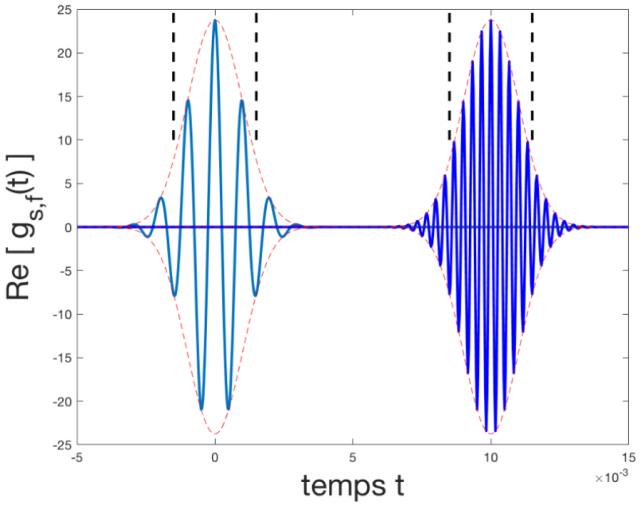
Example: Gabor limits





$$g_{s,f}(t) = w(t-s)e^{j2\pi ft}$$

$$w(t) = (\pi\sigma^2)^{-1/4}e^{-t^2/2\sigma^2}$$



Credits: Pierre Chainais, "De la transformée de Fourier à l'analyse temps-fréquence bivariée"

Heisenberg boxes

Time-frequency localization and spread

- Can we construct a function f, with an energy that is highly localized in time and with a Fourier transform $\widehat{f}$ having an energy concentrated in a small-frequency interval?
- To reduce the time spread of f, we can scale it by $a < 1$, while keeping its total energy constant:

$$f_a(t) = \frac{1}{\sqrt{a}} f\left(\frac{t}{a}\right), \quad \|f_a\|^2 = \|f\|^2$$

- The corresponding Fourier transform is dilated by a factor $1/a$:

$$\widehat{f_a}(\omega) = \sqrt{a}\,\widehat{f}(a\omega)$$

$\Rightarrow$ So we lose in frequency localization what we gained in time. Underlying is a trade-off between time and frequency localization.

Credits: S. Mallat (Wavelet tour)

Heisenberg's indeterminacy relations

Defining the average location and frequency respectively by:

$$b = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} t|f(t)|^2 \,\mathrm{d}t, \quad \xi = \frac{1}{2\pi\|f\|^2} \int_{-\infty}^{\infty} \omega|\widehat{f}(\omega)|^2 \,\mathrm{d}\omega$$

The variances around these average values are respectively:

$$\sigma_t^2 = \frac{1}{\|f\|^2} \int_{-\infty}^{\infty} (t-b)^2|f(t)|^2 \,\mathrm{d}t, \quad \sigma_\omega^2 = \frac{1}{2\pi\|f\|^2} \int_{-\infty}^{\infty} (\omega-\xi)^2|\widehat{f}(\omega)|^2 \,\mathrm{d}\omega$$

Theorem (Heisenberg's indeterminacy relations)

The temporal variance and the frequency variance of $f \in L^2(\mathbb{R})$ satisfy

$$\boxed{\sigma_t \, \sigma_\omega \geqslant \frac{1}{2}}$$

This inequality is an equality iff $\exists (b, \xi, c_1, c_2) \in \mathbb{R}^2 \times \mathbb{C}^2$ such that

$$f(t) = c_1 \mathrm{e}^{i\xi t - c_2(t-b)^2}$$

Credits: S. Mallat

Heisenberg's indeterminacy relations

Proof *(Weyl)*: this proof supposes that $\lim_{|t|\to+\infty} \sqrt{t} f(t) = 0$ $(*)$ but the theorem is valid for any $f \in L^2(\mathbb{R})$. The average time and frequency location of $\mathrm{e}^{-i\xi t} f(t+b)$ is zero. Thus, it is sufficient to prove the theorem for $b = \xi = 0$. Since $\widehat{f'(t)}(\omega) = i\omega \widehat{f}(\omega)$, the Plancherel identity applied to $i\omega\widehat{f}(\omega)$ yields

$$\sigma_t^2 \sigma_\omega^2 = \frac{1}{\|f\|^4} \left(\int_{-\infty}^{\infty} |t\, f(t)|^2 \,\mathrm{d}t \right) \left(\int_{-\infty}^{\infty} |f'(t)|^2 \,\mathrm{d}t \right) \quad (**)$$

Schwarz's inequality and the assumption $(*)$ [for the last equality] imply

$$\begin{aligned}
\sigma_t^2 \sigma_\omega^2 &\geqslant \frac{1}{\|f\|^4} \left(\int_{-\infty}^{\infty} |t\, f'(t) f^*(t)| \,\mathrm{d}t \right)^2 \qquad \forall z \in \mathbb{C},\ |z| \geqslant \mathrm{Re}(z) = \frac{z+z^*}{2} \\
&\geqslant \frac{1}{\|f\|^4} \left(\int_{-\infty}^{\infty} \frac{t}{2} (f'(t) f^*(t) + f'^*(t) f(t)) \,\mathrm{d}t \right)^2 \\
&\geqslant \frac{1}{4\|f\|^4} \left(\int_{-\infty}^{\infty} t (|f(t)|^2)' \,\mathrm{d}t \right)^2 \overset{\text{IBPF}}{=} \frac{1}{4\|f\|^4} \left(\int_{-\infty}^{\infty} |f(t)|^2 \,\mathrm{d}t \right)^2 = \frac{1}{4}
\end{aligned}$$

Credits: S. Mallat (Wavelet tour) (IBPF = Integration By Parts Formula)

Heisenberg's indeterminacy relations

Proof: To obtain an equality, Schwarz's inesquality applied to $(**)$ must be an equality. This implies that there exists $c_2 \in \mathbb{C}$ such that

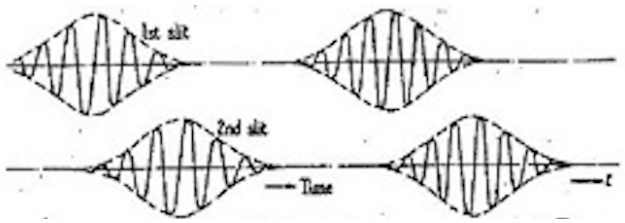
$$f'(t) = -2c_1 t f(t)$$

Thus, there exists $c_1 \in \mathbb{C}$ such that

$$f(t) = c_1 \mathrm{e}^{-c_2 t^2}$$

When $b \neq 0$ and $\xi \neq 0$ a time and frequency translation yield the result.

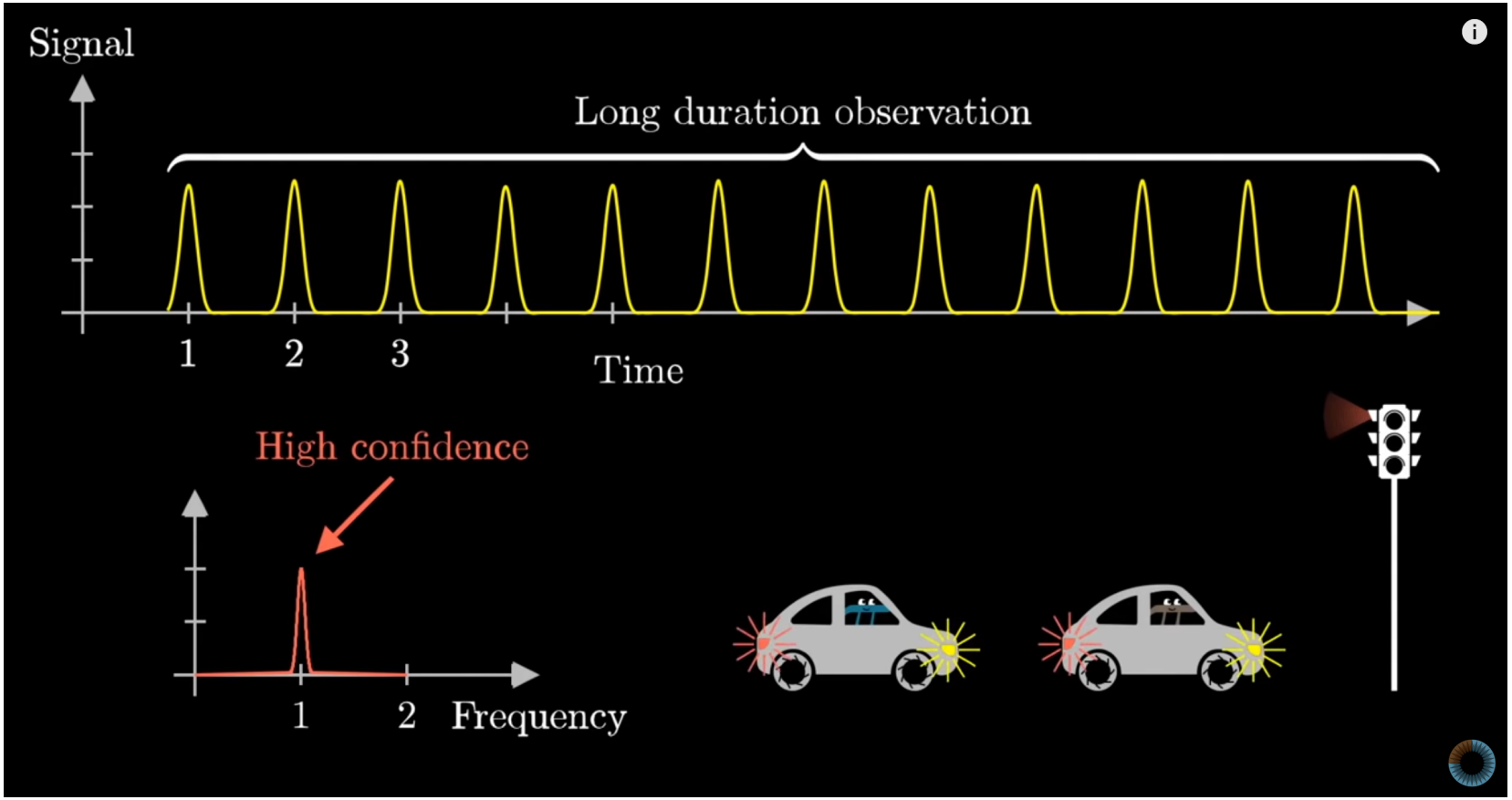
Remark: motivated by quantum mechanics, Gabor proposed time-frequency atoms that have a minimal spread in a time-frequency plane. By showing that signal decompositions over the dictionary of Gabor atoms are closely related to our perception of sounds, and that they exhibit important structures in speech and music recordings, he demonstrated the importance of localized time-frequency signal processing.



Credits: S. Mallat (Wavelet tour)

Heisenberg's indeterminacy relations

Some intuitions behind



$$\mathrm{TF}(f \cdot \Pi_{[-a/2,a/2]}) = \mathrm{TF}(f) * \mathrm{TF}(\Pi_{[-a/2,a/2]}) = \mathrm{TF}(f) * a\mathrm{sinc}(\pi a \cdot)$$

Credits: 3blue1brown (http://www.youtube.com/watch?v=MBnnXbOM5S4)

Heisenberg's indeterminacy relations

Some intuitions behind

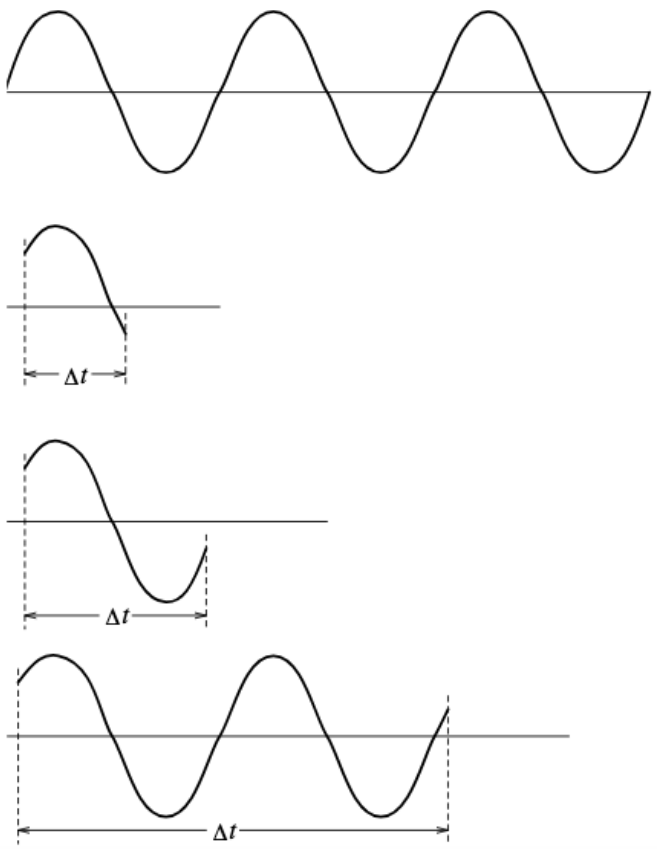


Figure: Improved frequency measurement over longer time intervals. The uncertainty in the frequency Δf decreases as the measurement interval Δt increases, and vice versa.

Credits: Bruce MacLennan (Gabor Representation)

Heisenberg's indeterminacy relations

Some intuitions behind

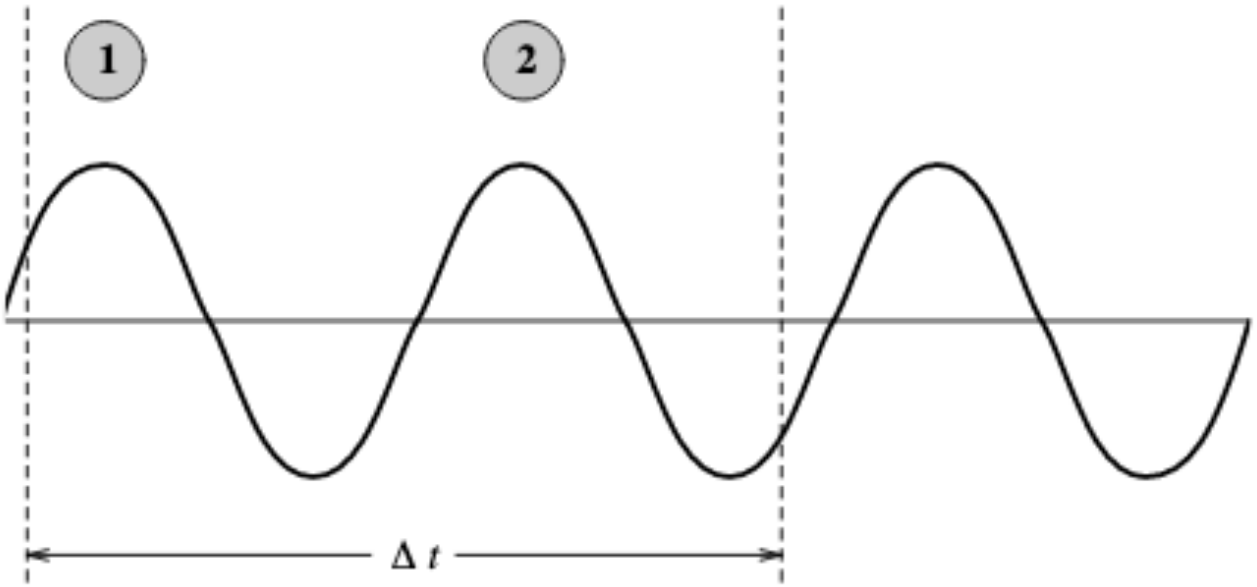


Figure: Measuring frequency by counting maxima in a given time interval. The circled numbers indicate the maxima counted during the measurement interval Δt. Since signals of other frequencies could also have the same number of maxima in that interval, there is an uncertainty Δf in the frequency.

Credits: Bruce MacLennan (Gabor Representation)

Heisenberg's indeterminacy relations

Some intuitions behind

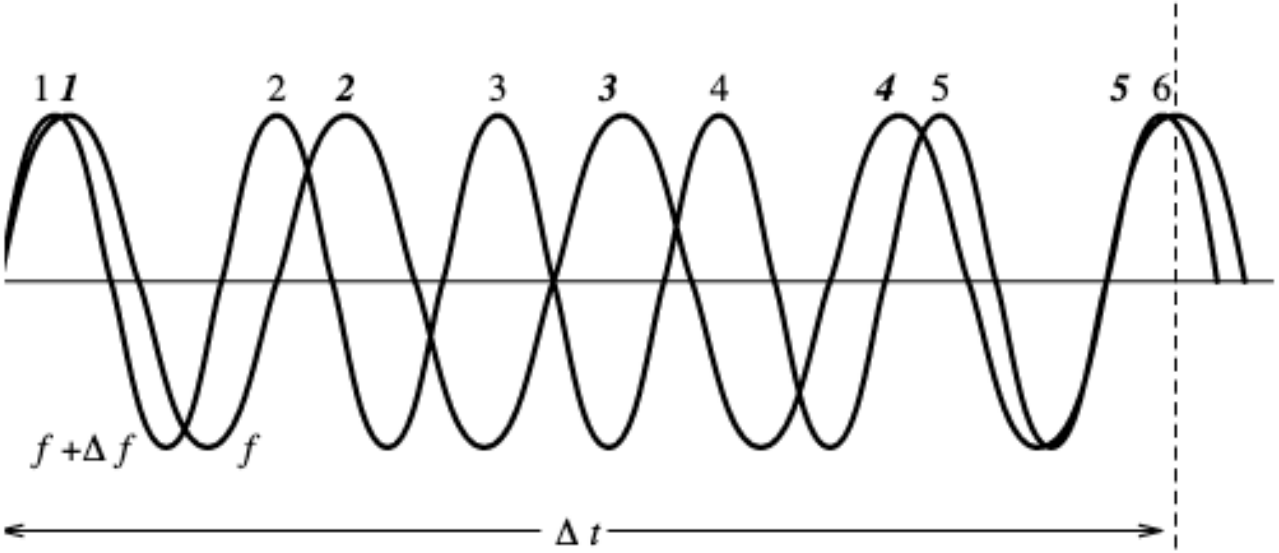


Figure: Minimum time interval Δt to detect frequency difference Δf. If two signals differ in frequency by Δf, then a measurement of duration $\Delta t \geqslant 1/\Delta f$ is required to guarantee a difference in counts of maxima. (Italic numbers indicate maxima of signal of frequency f, roman numbers indicate maxima of signal of higher frequency $f + \Delta f$)

$$(f + \Delta f)\Delta t - f\Delta t \geqslant 1 \Leftrightarrow \Delta f\, \Delta t \geqslant 1$$

Credits: Bruce MacLennan (Gabor Representation)

Short Time Fourier Transform

Examples

1. A sinusoidal wave $f(t) = \mathrm{e}^{i\xi_0 t}$ whose Fourier transform is a Dirac $\widehat{f}(\omega) = 2\pi\delta(\omega - \xi_0)$ has a STFT:

$$Sf(\xi, b) = \widehat{w}(\xi - \xi_0)\mathrm{e}^{-ib(\xi-\xi_0)}$$

   Its energy is spread over the frequency interval

$$\xi \in [\xi_0 - \sigma_\omega/2, \xi_0 + \sigma_\omega/2]$$

2. A Dirac $f(t) = \delta(t - b_0)$ has a STFT:

$$Sf(\xi, b) = w(b - b_0)\mathrm{e}^{-i\xi b_0}$$

   Its energy is spread in the time interval

$$b \in [b_0 - \sigma_t/2, b_0 + \sigma_t/2]$$

Limitation of the Short Time Fourier Transform

The STFT cannot separate events of a distance smaller than a_0, that is to localize the two frequencies and the transient phenomena.

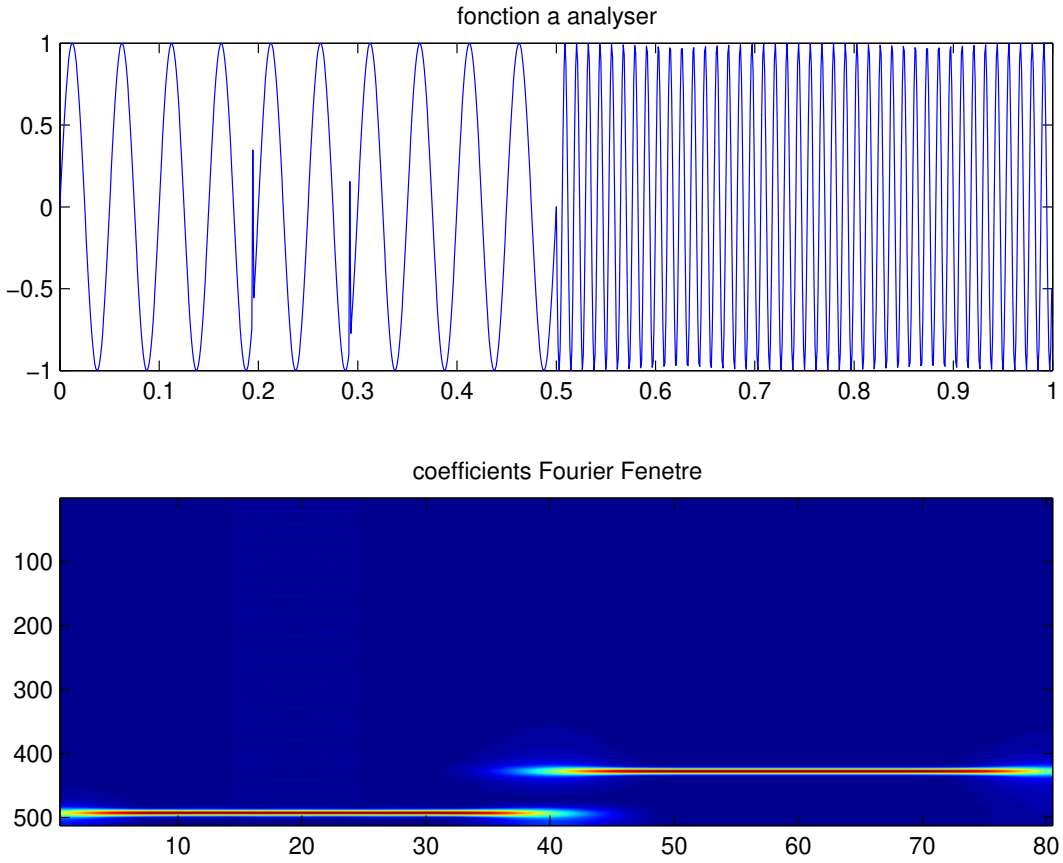


Figure: Signal $f_2 = f_1 + \delta_1 + \delta_2$ and its Gabor transform with $a_0 = 0.05$

Limitation of the Short Time Fourier Transform

The STFT cannot separate events of a distance smaller than a_0, that is to localize the two frequencies and the transient phenomena.

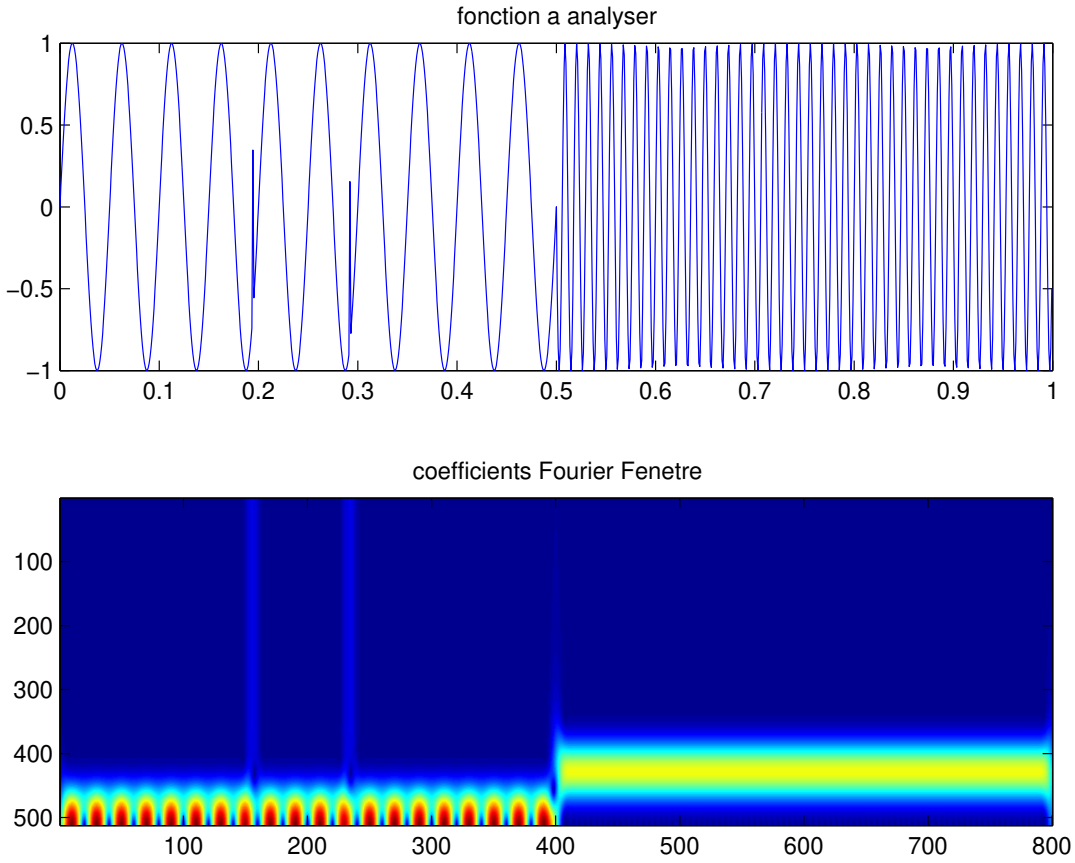
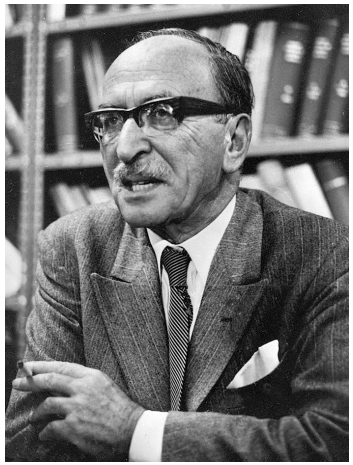


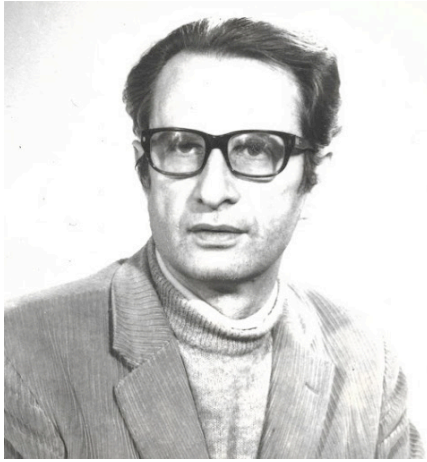
Figure: Signal $f_2 = f_1 + \delta_1 + \delta_2$ and its Gabor transform with $a_0 = 0.005$

Pioneer works on wavelets

- **Jean Morlet** research engineer at ELF Aquitaine discovered wavelets for solving signal processing problems arising from oil exploration.
- **Alex Grossmann** recognized in the Morlet wavelets something similar to coherent states formalism in quantum mechanics and developed an exact inversion formula for the wavelet transform.
- They developed the mathematics of the continuous wavelet transforms in their article: "Decomposition of Hardy Functions into Square Integrable Wavelets of Constant Shape" (1984)



Dennis Gabor



Alex Grossmann

Jean Morlet

"Gaborettes" vs Morlet wavelets

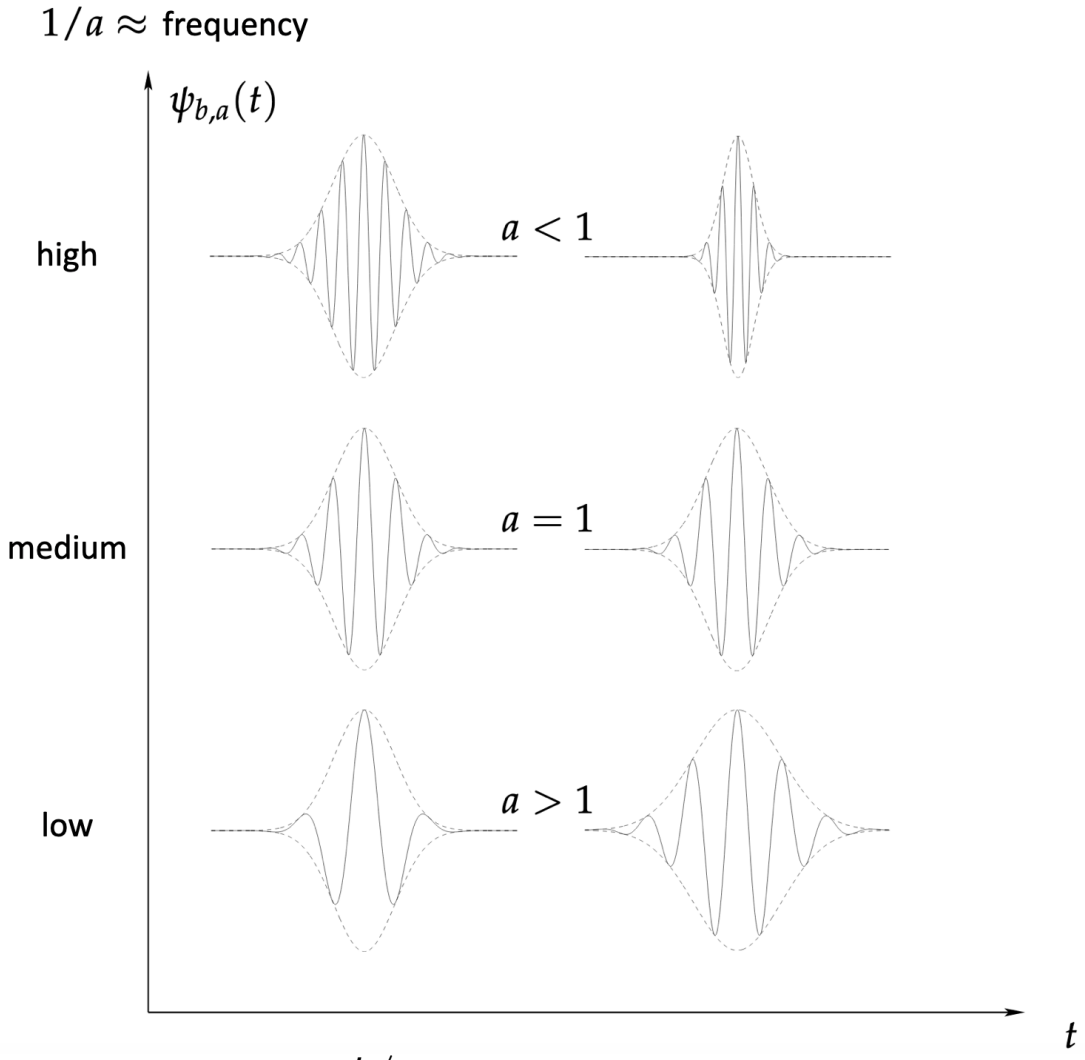
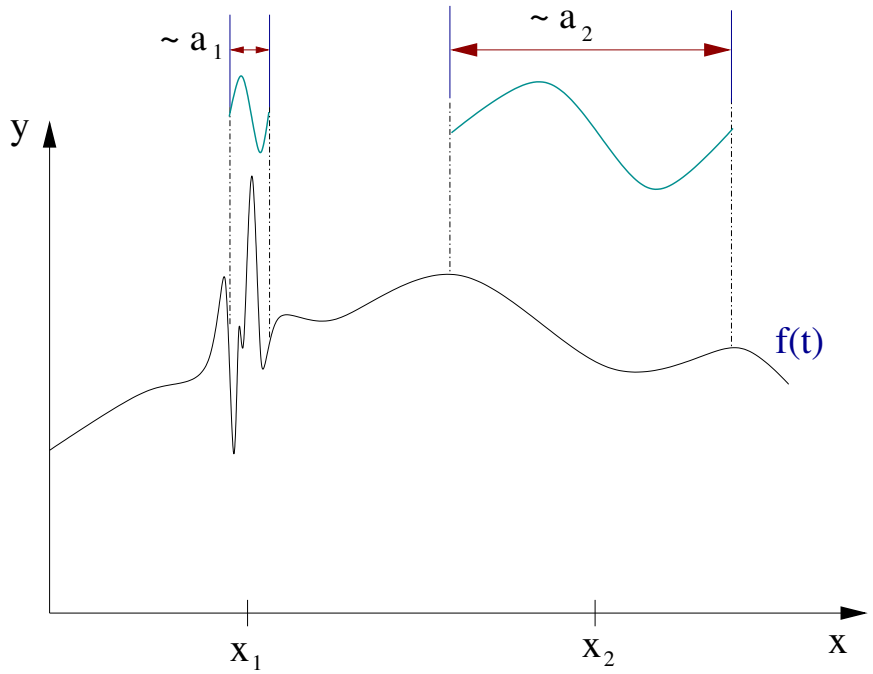


Figure: (Left) Gabor $\psi_{b,a}(t) = \mathrm{e}^{it/a}\psi(t-b)$, (right) Morlet $\psi_{b,a}(t) = a^{-1/2}\psi\left(\frac{t-b}{a}\right)$

Gabor $\Rightarrow$ frequency modulation inside a constant window width

Wavelets $\Rightarrow$ shape of $\psi_{b,a}$ doesn't change, simply dilated or compressed

The Continuous Wavelet Transform (CWT) – Definition



$$Wf(a,b) = \int_{-\infty}^{+\infty} f(x)\overline{\psi_{a,b}(x)}\,\mathrm{d}x = \langle f, \psi_{\nu,b}\rangle, \quad a > 0,\ b \in \mathbb{R}$$

The analyzing functions or **wavelets** are defined by:

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)$$

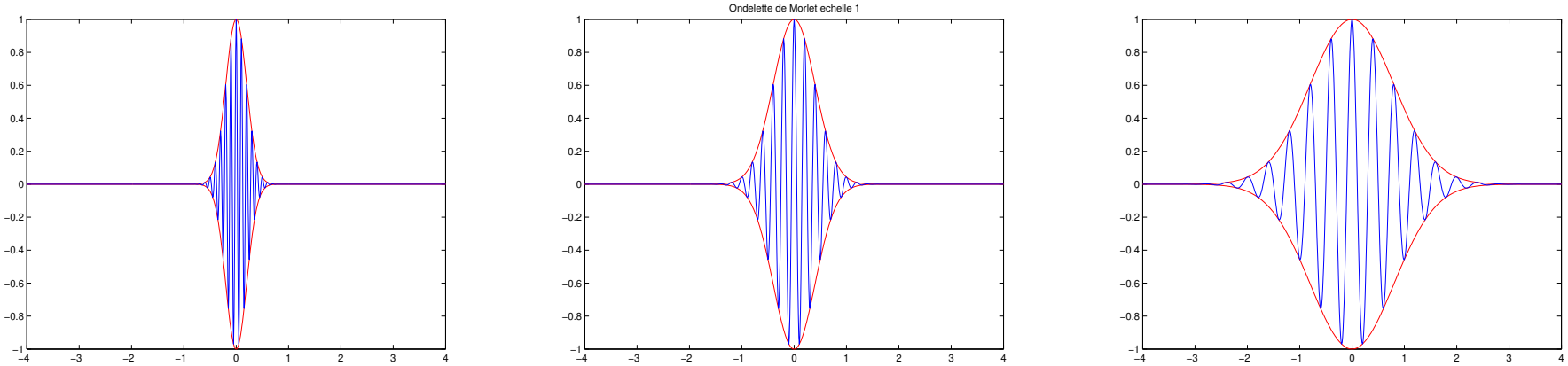
Wavelet family in physical space

Example: the Morlet wavelets

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right)$$

with mother wavelet

$$\psi(x) = \cos(x)\mathrm{e}^{-10\pi x^2}$$



(a) $a = 1/2$ (b) $a = 1$ (c) $a = 2$

Figure: Morlet wavelets of scale: $a = 1/2,\ 1,\ 2$ (real part). The scale a gives the support size (inverse of a frequency), whereas b gives the position.

Wavelet analysis of the toy signal with Morlet wavelets

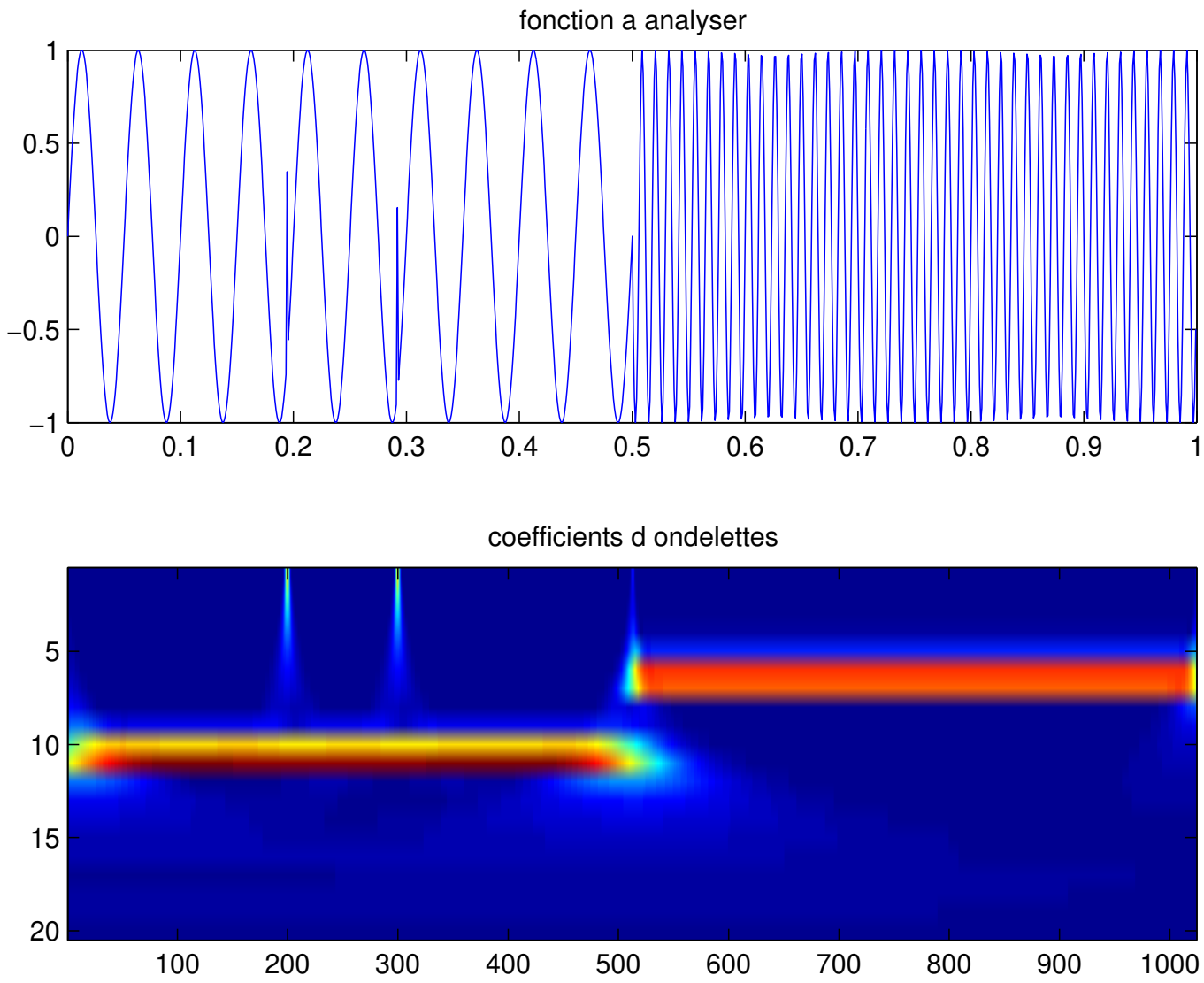


Figure: Signal f_2 (two notes + scratch) and its CWT

Wavelet definition

A function $\psi(x) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ is a **wavelet** if it satisfies the following *admissiblity condition*:

$$C_\psi = \int_{-\infty}^{+\infty} \frac{\left|\widehat{\psi}(\nu)\right|^2}{|\nu|} \, \mathrm{d}\nu \; < \; \infty$$

which implies $\int_{-\infty}^{+\infty} \psi(x) \, \mathrm{d}x = 0$ (and this is equivalent if $x\psi$ integrable).

Examples

1. The (complex) Morlet wavelet
   - Mother wavelet: $\psi(x) = \mathrm{e}^{-\pi x^2} \mathrm{e}^{10 i \pi x}$
   - Its Fourier Transform: $\hat{\psi}(\nu) = \mathrm{e}^{-\pi(\nu-5)^2}$
2. Gaussian derivatives
   - Mother wavelet: $\psi_n(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n} \mathrm{e}^{-\pi x^2}, \quad n \geq 1$
     (for $n = 2$, ψ_2 is called the "Mexican Hat")
   - Its Fourier Transform: $\hat{\psi}_n(\nu) = (2 i \pi \nu)^n e^{-\pi \nu^2}$

Wavelet analysis

A picture is worth a thousand words

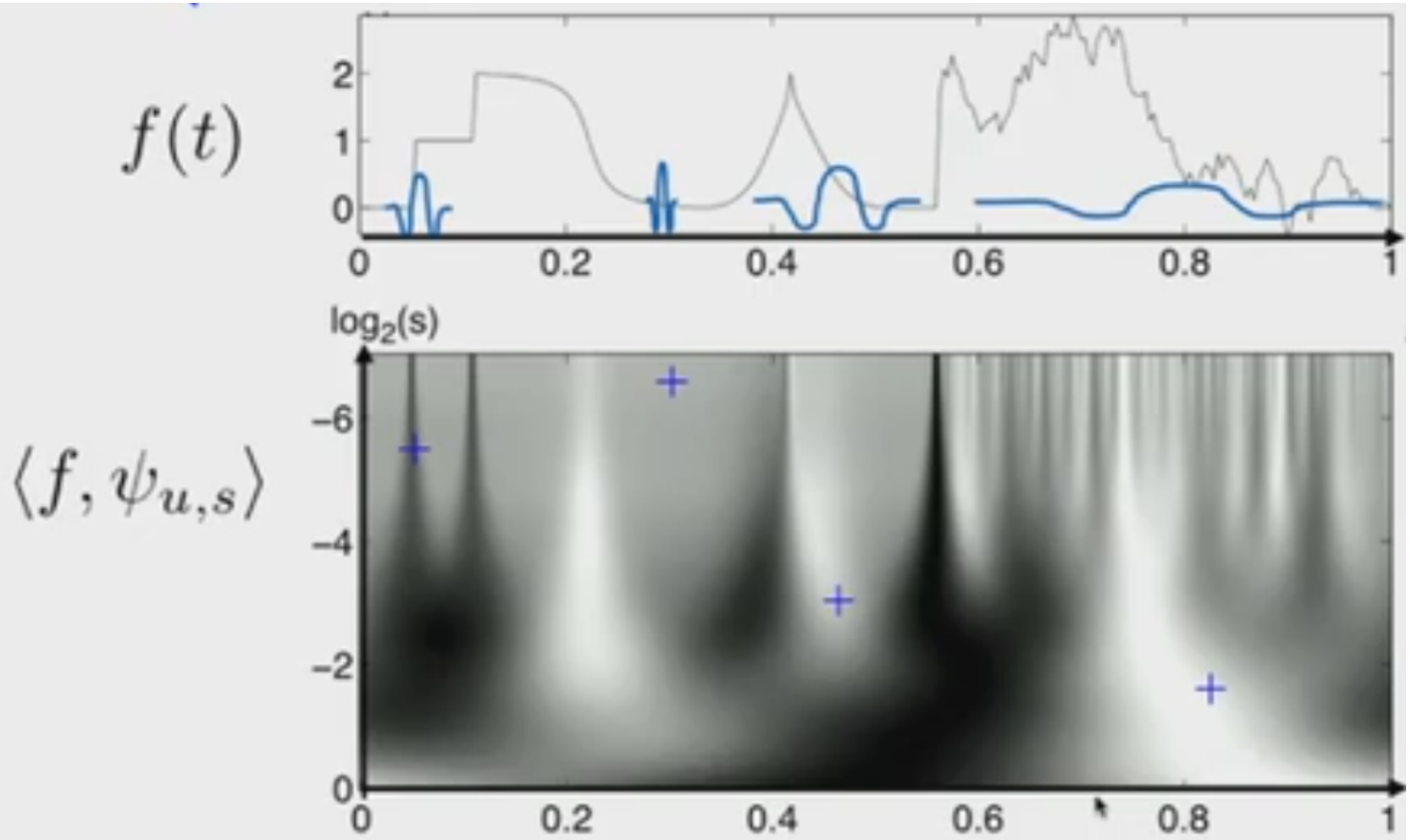


Figure: Correlations with Morlet wavelets translated and dilated

Fourier Transform of wavelets

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right) \longleftrightarrow \hat{\psi}_{a,b}(\nu) = \sqrt{a}\hat{\psi}(a\nu)\mathrm{e}^{-2i\pi b\nu}$$

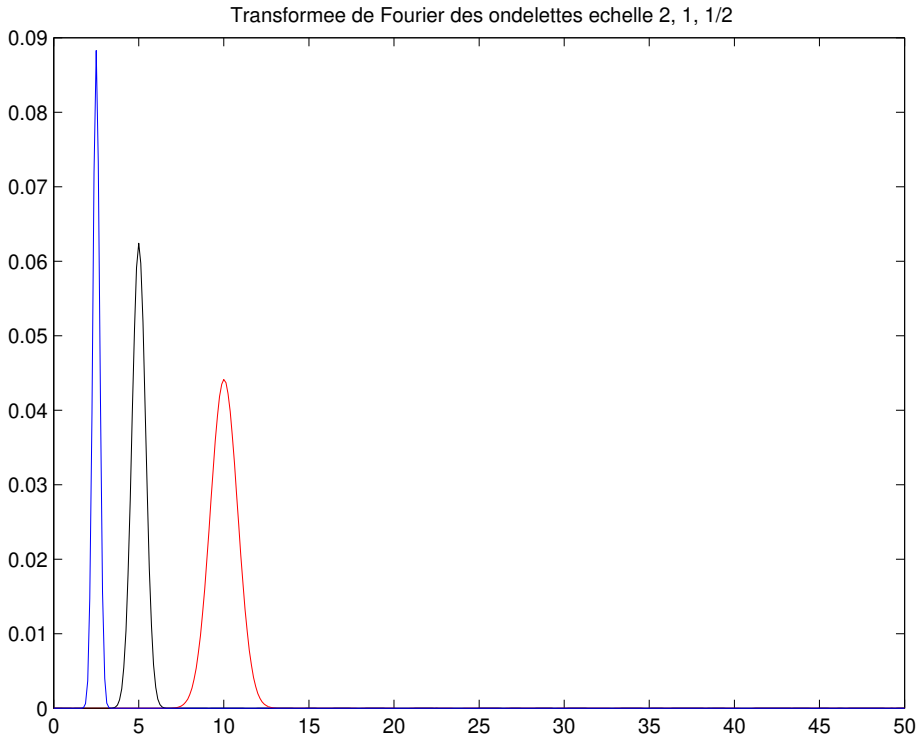


Figure: Fourier Transform (modulus) of Morlet wavelets of scales $a = 1/2,\ 1,\ 2$. Wavelets behaves as band-pass filters around frequency $\nu = \frac{\nu_0}{a}$, where ν_0 is the peak wavenumber (max of $\hat{\psi}$). For the Morlet wavelet, $\nu_0 = 5$.

Equivalent definition

Let $f \in L^2(\mathbb{R})$. Forall $a > 0, \quad b \in \mathbb{R}$,

$$\begin{aligned} Wf(a,b) &= \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} \, \mathrm{d}x \\ Wf(a,b) &= \sqrt{a} \int_{-\infty}^{+\infty} \hat{f}(\nu) \overline{\hat{\psi}(a\nu)} \mathrm{e}^{2i\pi\nu b} \, \mathrm{d}\nu \end{aligned}$$

Proof: *From the Parseval formula*

$$Wf(a,b) = \langle f, \psi_{a,b} \rangle = \langle \hat{f}, \hat{\psi}_{a,b} \rangle$$

- In the time domain (x), $Wf(a,b)$ provides information on the signal f around point b in a vicinity of size $\sim a$.
- In the frequency domain (ν), $Wf(a,b)$ provides information on the signal $\hat{f}$ around frequency $\sim \frac{1}{a}$.

$\Rightarrow$ Wavelet analysis is a time-scale analysis

Time-frequency resolution of wavelets

$$\psi_{a,b}(x) = \frac{1}{\sqrt{a}}\psi\left(\frac{x-b}{a}\right) \longleftrightarrow \hat{\psi}_{a,b}(\nu) = \sqrt{a}\hat{\psi}(a\nu)\mathrm{e}^{-2i\pi b\nu}$$

We suppose that ψ is analytic, $\psi(0) = 0$ and $\eta = \frac{1}{2\pi}\int_0^\infty \omega|\widehat{\psi}(\omega)|^2\,\mathrm{d}\omega$

$$\int_{-\infty}^{\infty}(t-b)^2|\psi_{a,b}(t)|^2\,\mathrm{d}t \overset{t\leftarrow\frac{t-b}{a}}{=} \int_{-\infty}^{\infty} a^2t^2|\psi(t)|^2\,\mathrm{d}t = a^2\sigma_t^2$$

$$\frac{1}{2\pi}\int_0^\infty\left(\omega-\frac{\eta}{a}\right)^2|\widehat{\psi}_{\xi,b}(\omega)|^2\,\mathrm{d}\omega = \frac{1}{2\pi a^2}\int_0^\infty(\omega-\eta)^2|\widehat{\psi}(\omega)|^2\,\mathrm{d}\omega = \frac{\sigma_\omega^2}{a^2}$$

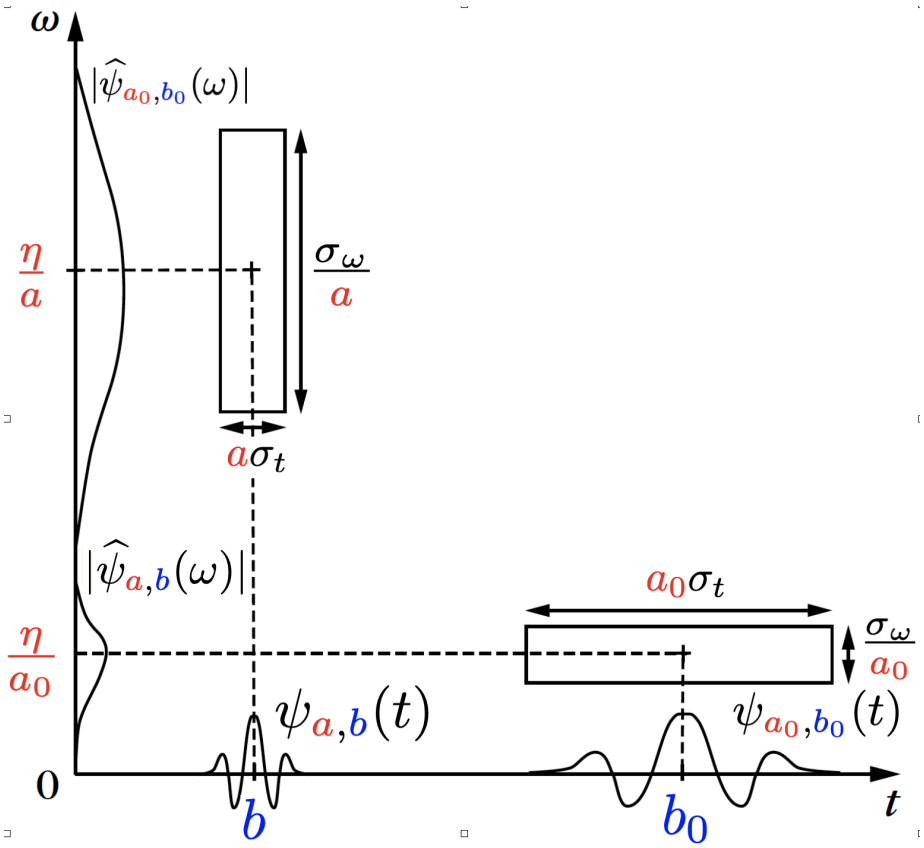
The energy spread of a wavelet time-frequency atom $\psi_{a,b}$ corresponds to a Heisenberg box centered at $(b, \xi = \eta/a)$, of size $a\,\sigma_t$ along time and σ_ω/a along frequency. The area of the rectangle remains equal to $\sigma_t\sigma_\omega$ at all scales but the resolution in time and frequency depends on a. An analytic wavelet transform defines a local time-frequency energy density

$$P_W f(b,\xi) = |Wf(a,b)|^2 = \left|Wf\left(\frac{\eta}{\xi}, b\right)\right|^2 \quad (\textit{scalogram})$$

Heisenberg boxes of two wavelets $\psi_{a,b}$ and ψ_{a_0,b_0}

$$\int_{-\infty}^{\infty} (t-b)^2 |\psi_{a,b}(t)|^2 \, \mathrm{d}t = a^2 \sigma_t^2$$

$$\frac{1}{2\pi} \int_0^{\infty} \left(\omega - \frac{\eta}{a}\right)^2 |\widehat{\psi}_{\xi,b}(\omega)|^2 \, \mathrm{d}\omega = \frac{\sigma_\omega^2}{a^2}$$



Inversion of the Continuous Wavelet Transform

Synthesis formula and energy conservation

Theorem (Calderón, Grossmann and Morlet)

Let $\psi \in L^2(\mathbb{R})$ be a real function such that

$$C_\psi = \int_{-\infty}^{+\infty} \frac{\left|\widehat{\psi}(\nu)\right|^2}{|\nu|} \, \mathrm{d}\nu \; < \; \infty$$

Any $f \in L^2(\mathbb{R})$ satisfies

$$f(x) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} Wf(a,b) \psi_{a,b}(x) \frac{\mathrm{d}a \, \mathrm{d}b}{a^2} \; (*)$$

and

$$\int_{-\infty}^{+\infty} |f(x)|^2 \, \mathrm{d}x = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} \left|Wf(a,b)\right|^2 \frac{\mathrm{d}a \, \mathrm{d}b}{a^2} \; (**)$$

Proof *(Synthesis formula):* For a fixed a, the CWT can be written:

$$Wf(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x)\, \overline{\psi\left(\frac{x-b}{a}\right)}\, \mathrm{d}x = (f * \check{\psi}_a)(b)$$

where we have noted:

$$\psi_a(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x}{a}\right), \quad \check{\psi}_a(x) = \psi_a(-x)$$

The right integral $b(x)$ of $(*)$ can now be rewritten as a sum of convolutions:

$$b(x) = \frac{1}{C_\psi} \int_0^{+\infty} Wf(a,\cdot) * \psi_a(x) \frac{\mathrm{d}a}{a^2} = \frac{1}{C_\psi} \int_0^{+\infty} f * \check{\psi}_a * \psi_a(x) \frac{\mathrm{d}a}{a^2}$$

$$\widehat{b}(\omega) = \frac{1}{C_\psi} \int_0^{+\infty} \widehat{f}(\omega) \sqrt{a}\, \widehat{\psi}^*(a\,\omega)\, \sqrt{a} \widehat{\psi}(a\,\omega) \frac{\mathrm{d}a}{a^2} = \frac{\widehat{f}(\omega)}{C_\psi} \int_0^{+\infty} |\widehat{\psi}(a\,\omega)|^2 \frac{\mathrm{d}a}{a^2}$$

By the change of variable $\xi = a\,\omega$ we get $\widehat{b}(\omega) = \frac{\widehat{f}(\omega)}{C_\psi} \int_0^{+\infty} \frac{|\widehat{\psi}(\xi)|^2}{\xi}\, \mathrm{d}\xi = \widehat{f}(\omega)$

The equality of their Fourier transform leads to $b = f$. QED $\square$

Inversion with a different synthesis wavelet

- **Decomposition** with an *analysing wavelet* g: $a > 0$, $b \in \mathbb{R}$,

$$W_g f(a,b) = \int_{-\infty}^{+\infty} f(x) \frac{1}{\sqrt{a}} \bar{g}\left(\frac{x-b}{a}\right) \mathrm{d}x$$

- **Synthesis** with a *reconstruction wavelet* h:

$$f(x) = \frac{2}{c_{gh}} \int_0^{+\infty} \int_{-\infty}^{+\infty} W_g f(a,b) \, \frac{1}{\sqrt{a}} h\left(\frac{x-b}{a}\right) \, \frac{\mathrm{d}a}{a^2} \, \mathrm{d}b$$

- **Cross-admissibility condition** on wavelets g et h ($g, h \in L^2(\mathbb{R})$):

$$c_{gh} = \int_{-\infty}^{+\infty} \frac{\bar{\hat{g}}(k)\hat{h}(k)}{|k|} \, \mathrm{d}k < +\infty$$

Remark: In this case, only h or g has to be a zero mean function.

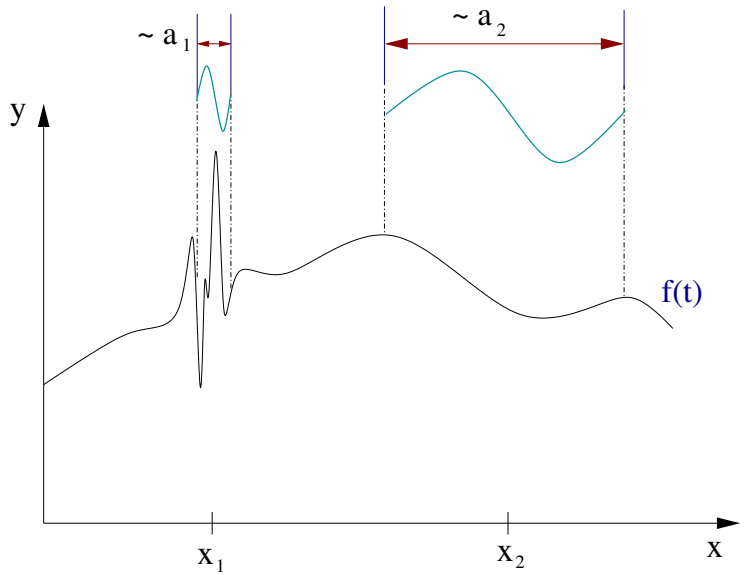
Coding – In practice

For a fixed a, the CWT is a convolution product:

$$\begin{aligned} Wf(a,b) &= \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x)\, \overline{\psi\left(\frac{x-b}{a}\right)}\, \mathrm{d}x \\ &= (f * \check{\psi}_a)(b) \end{aligned}$$

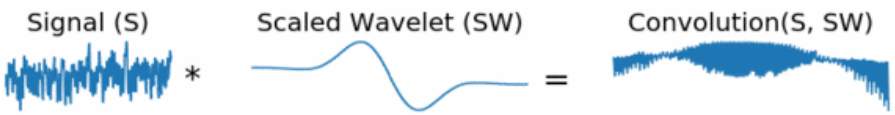
where we have noted:

$$\check{\psi}_a(x) = \frac{1}{\sqrt{a}} \psi\left(-\frac{x}{a}\right)$$

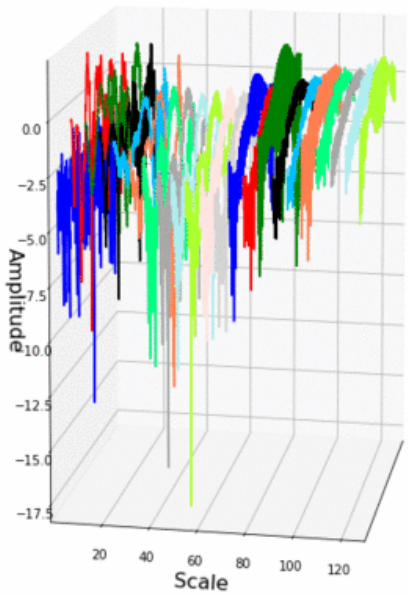


Coding – In practice

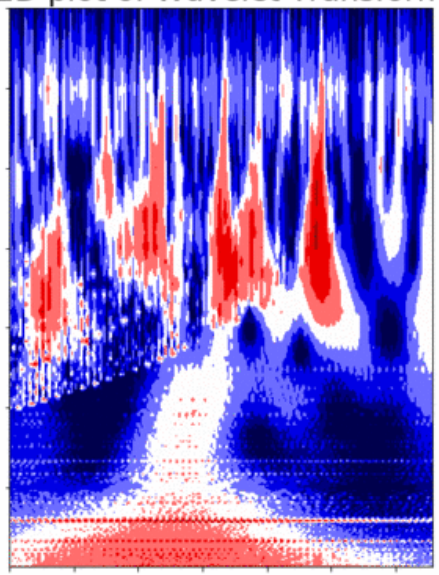
$$f * \check{\psi}_a : b \mapsto Wf(a, b)$$



3D plot of Wavelet Transform



2D plot of Wavelet Transform



source: ataspinar.com

Examples and Interpretation

Example 1: pure cosine

If f is a **pure cosine** $f(x) = \cos(2\pi k x)$, then

$$\begin{aligned} Wf(a,b) &= \int_{-\infty}^{+\infty} \left(\frac{\mathrm{e}^{2i\pi kx} + \mathrm{e}^{-2i\pi kx}}{2} \right) \overline{\psi_{a,b}(x)} \ \mathrm{d}x \\ &= \frac{1}{2} \left[\overline{\hat{\psi}_{a,b}(k)} + \overline{\hat{\psi}_{a,b}(-k)} \right] \\ &= \frac{\sqrt{a}}{2} \left[\overline{\hat{\psi}(ak)} \mathrm{e}^{2i\pi kb} + \overline{\hat{\psi}(-ak)} \mathrm{e}^{-2i\pi kb} \right] \end{aligned}$$

- If the wavelet ψ is analytic complex:

$$Wf(a,b) = \frac{\sqrt{a}}{2} \overline{\hat{\psi}(ak)} \mathrm{e}^{2i\pi kb}$$

- If the wavelet ψ is real $Wf(a,b) = \sqrt{a} \ \mathrm{Re} \left(\hat{\psi}(ak) \ e^{-2i\pi kb} \right)$

Example 1: pure cosine

Pure cosine $f(x) = \cos(20\pi x)$ ($k = 10$)

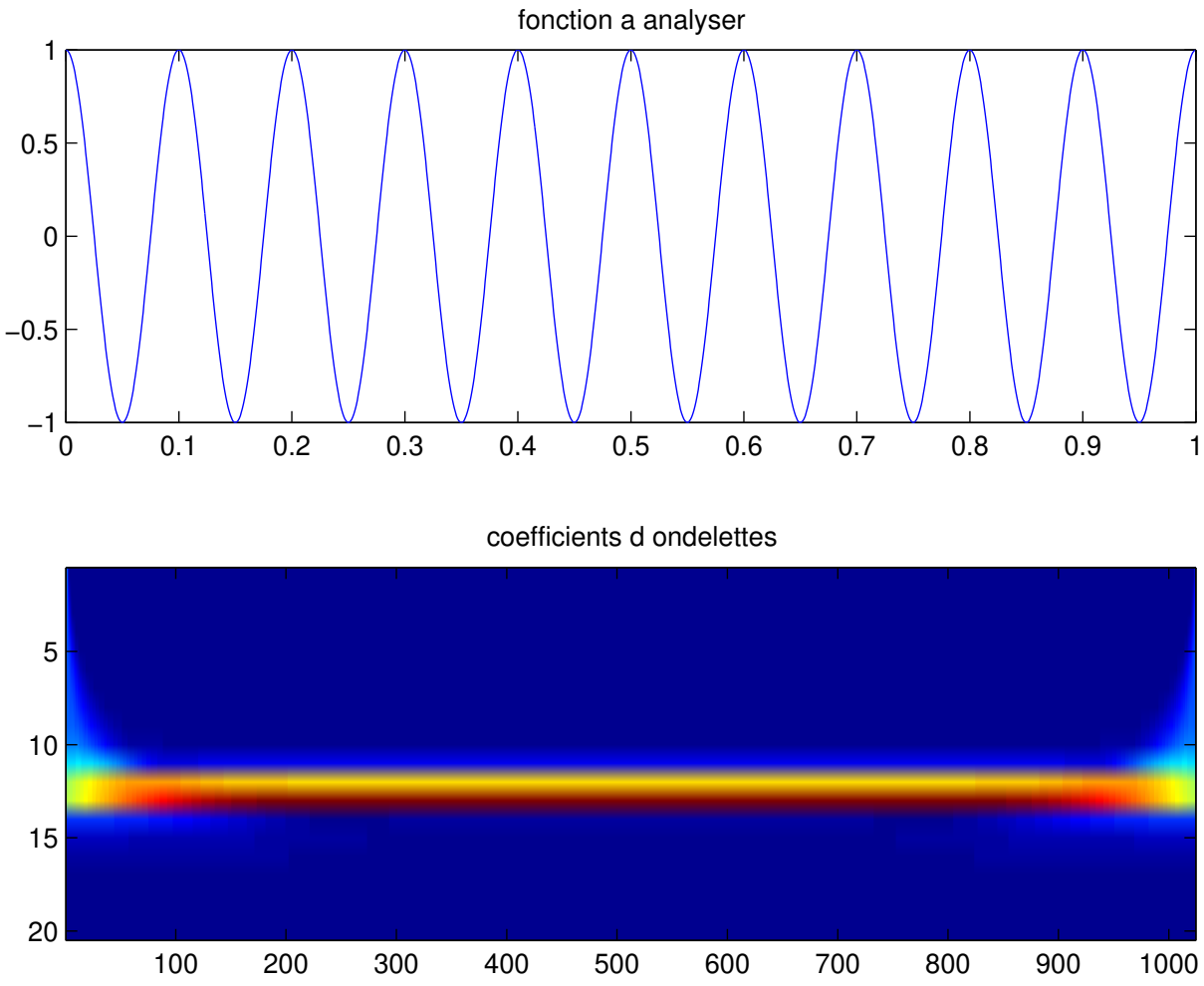


Figure: CWT (modulus), using the Morlet wavelet (analytic complex)

Example 1: pure cosine

Pure cosine $f(x) = \cos(20\pi x)$ ($k = 10$)

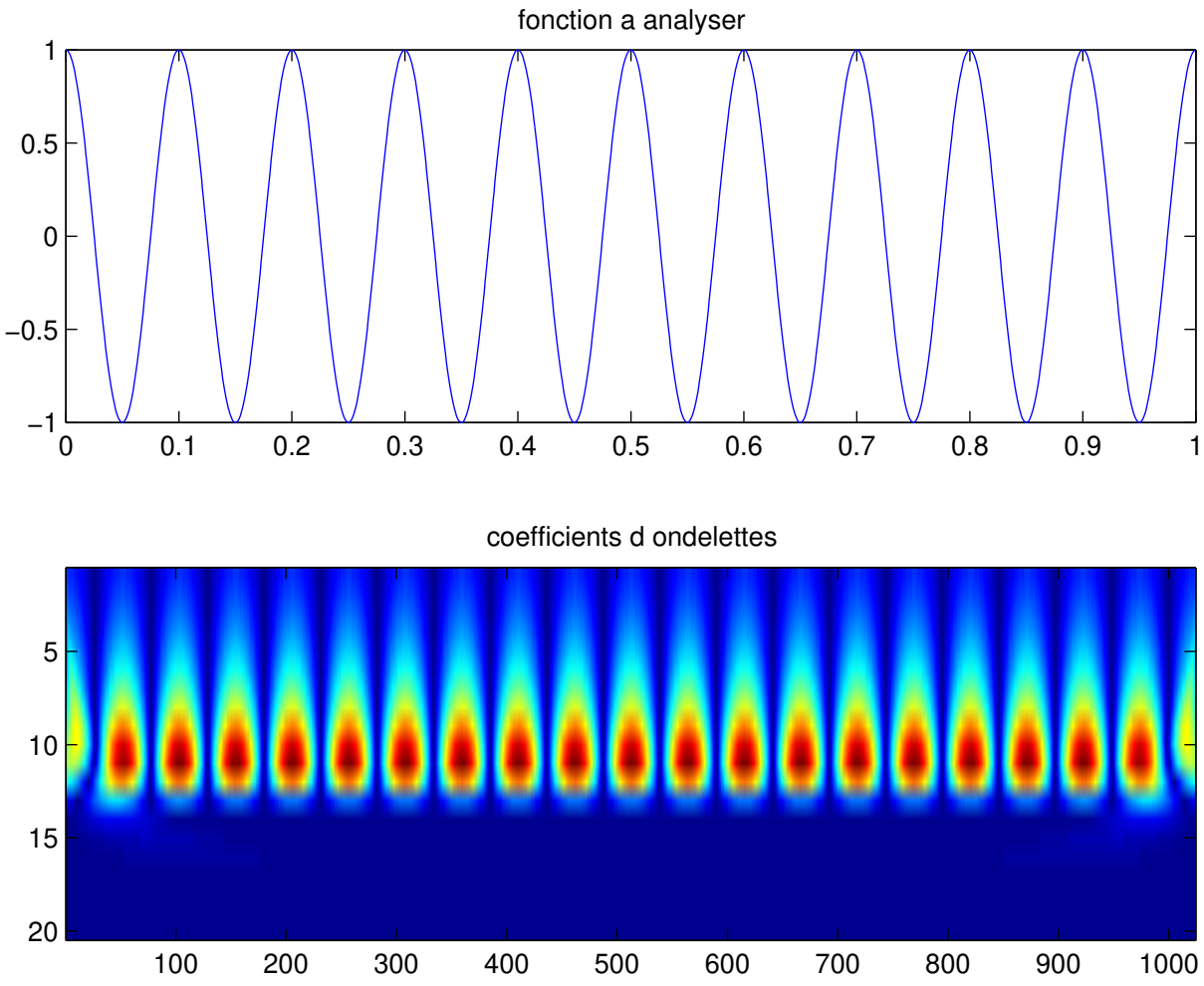


Figure: CWT (modulus), using the Gaussian derivatives (real)

Examples and Interpretation

Example 2: a Dirac

If f is a **Dirac** $f(x) = \delta(x - x_0)$ (pointwise measure supported by x_0), then:

$$\begin{aligned} Wf(a,b) &= \int_{-\infty}^{+\infty} \delta(x - x_0)\, \overline{\psi_{a,b}(x)}\, \mathrm{d}x \\ &= \overline{\psi_{a,b}(x_0)} \\ &= \frac{1}{\sqrt{a}}\, \overline{\psi\left(\frac{x_0 - b}{a}\right)} \end{aligned}$$

Remark: At each scale a, $b \to Wf(a,b)$ is the wavelet of scale a centered on x_0 (up to a symmetry).

Example 2: a Dirac

Dirac δ_{x_0}

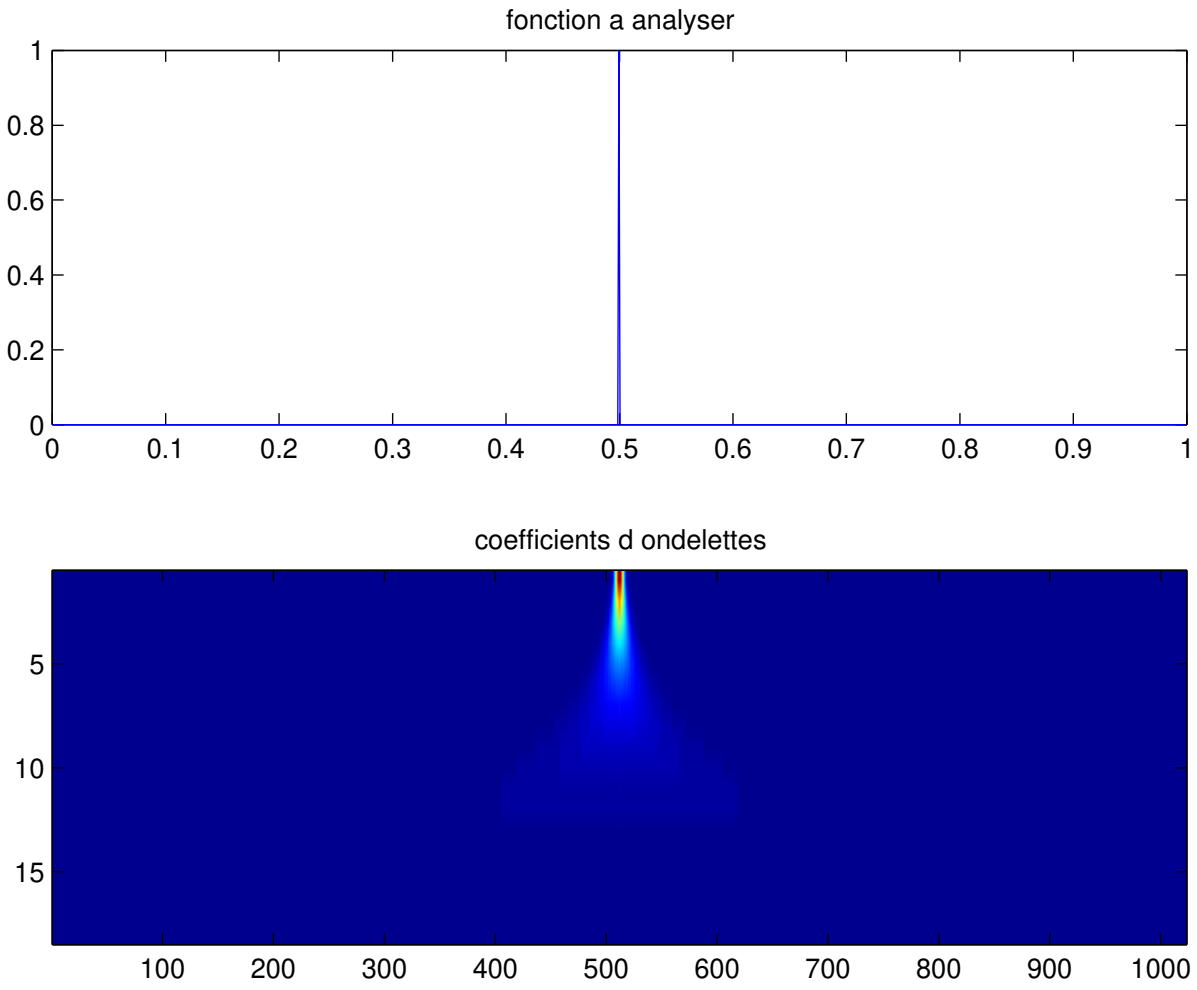


Figure: Signal "Dirac" and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)

Example 3: the periodic square wave

Periodic square wave

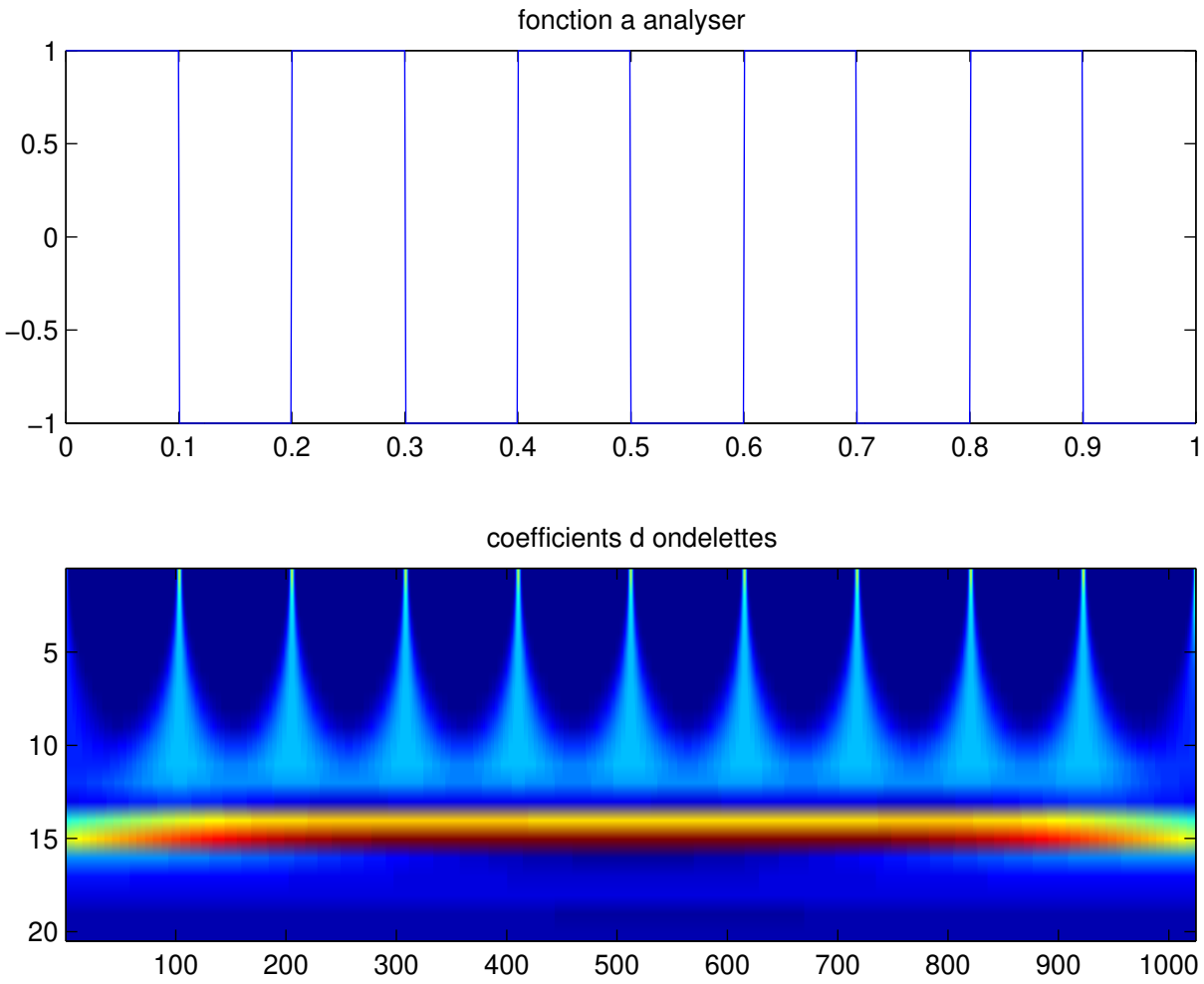


Figure: Square wave and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)

Example 4: a modulated wave

Modulated wave

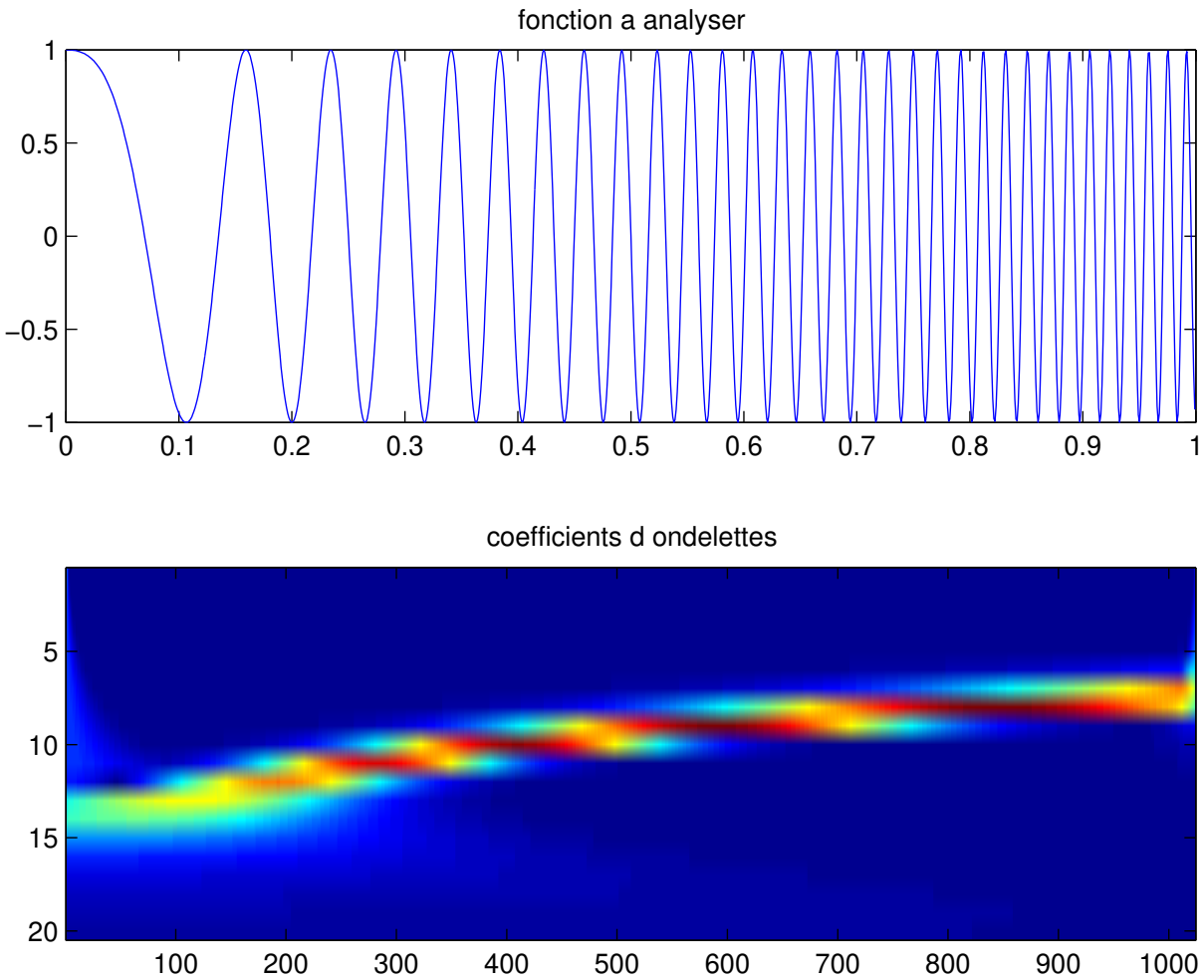


Figure: Modulated wave and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)

Example 5: 2 sinusoids with noise

2 sinusoids with noise

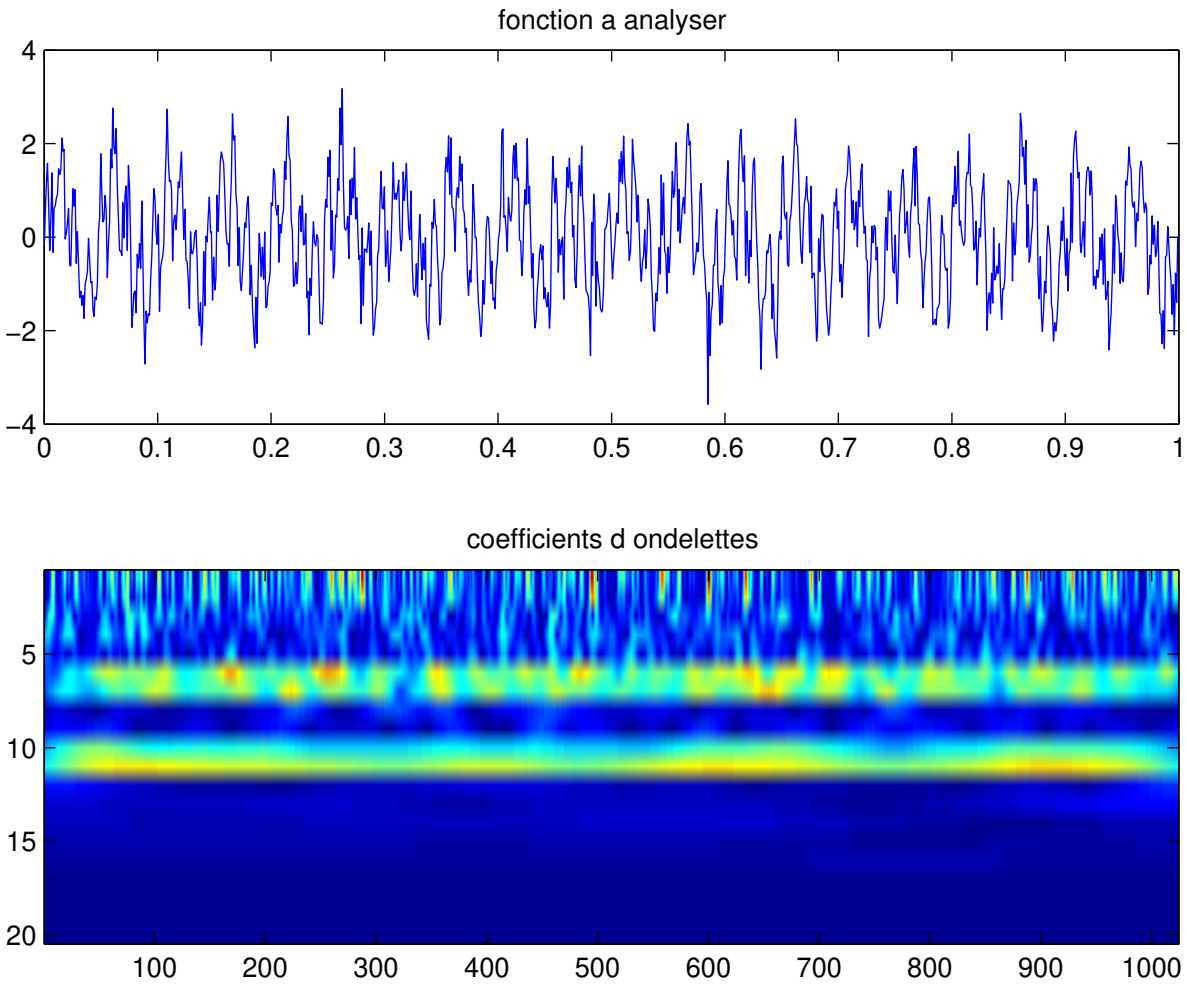


Figure: Signal and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)

Example 6: Holder function of exponant $\frac{1}{2}$

Holder function of exponant $\frac{1}{2}$

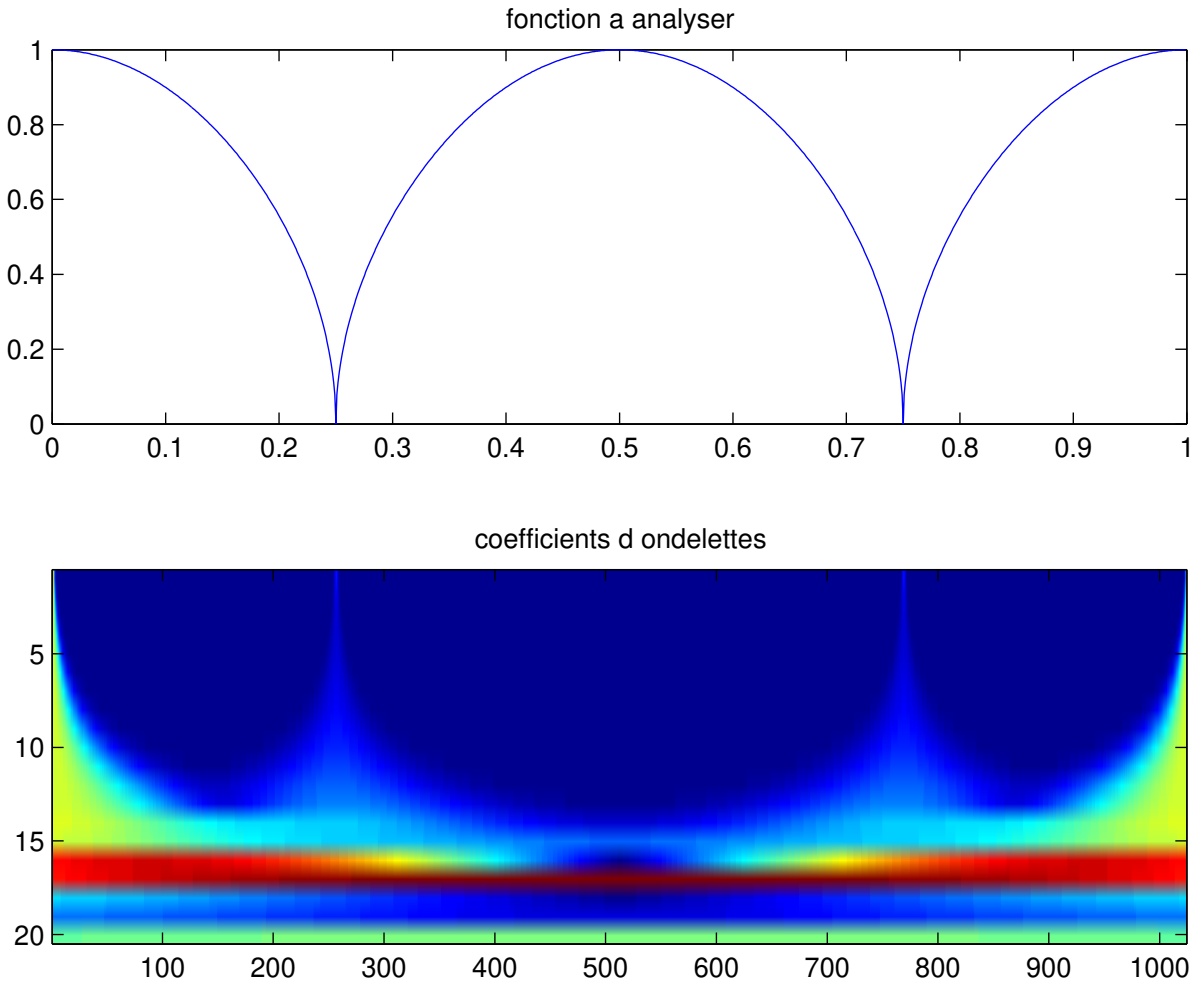
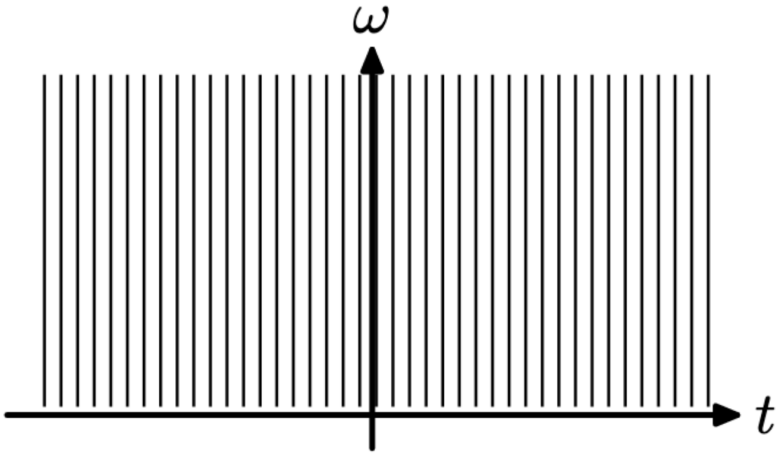


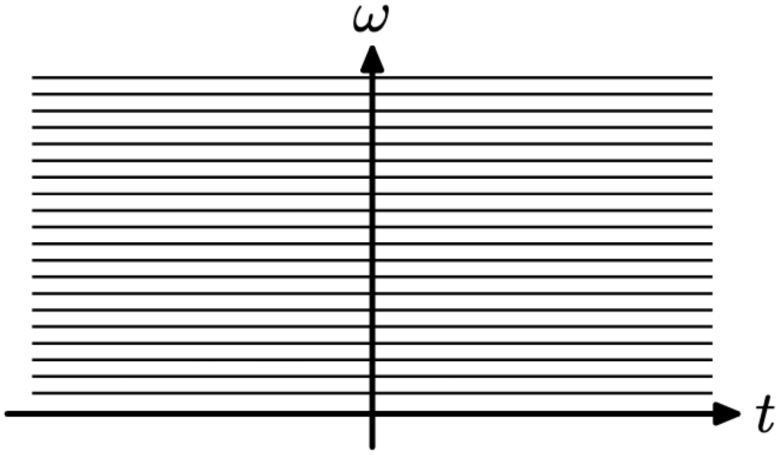
Figure: $f(x) = \sqrt{|\cos(2\pi x)|}$ and its CWT (modulus, Morlet wavelet, divided by $\sqrt{a}$)

Take home message

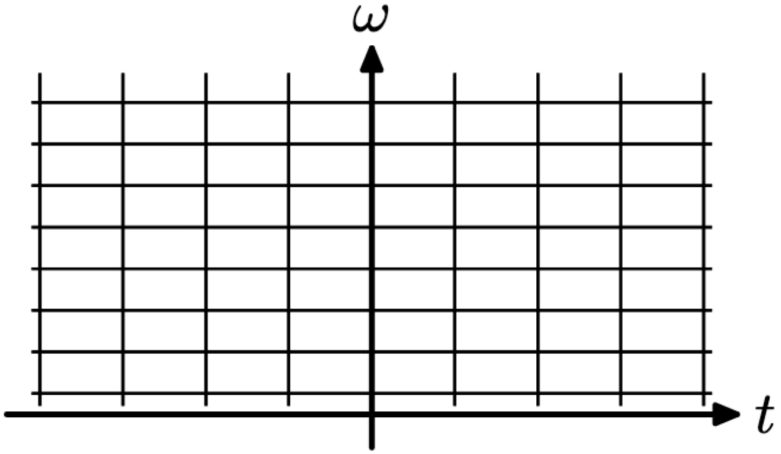
Time-frequency vs time-scale



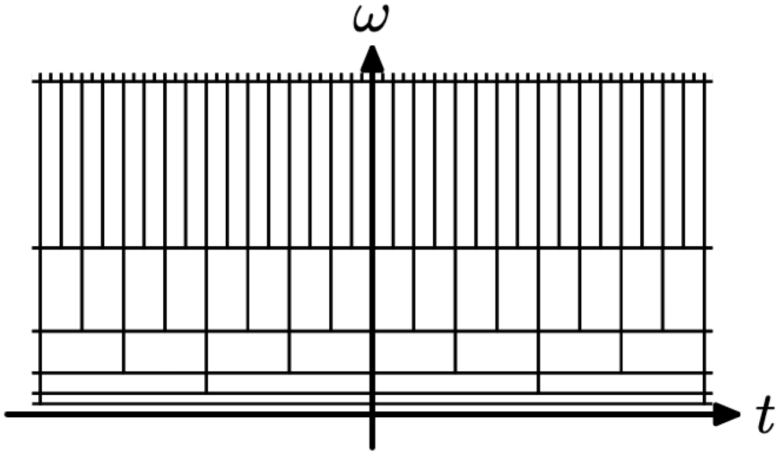
(a) Diracs



(b) Fourier



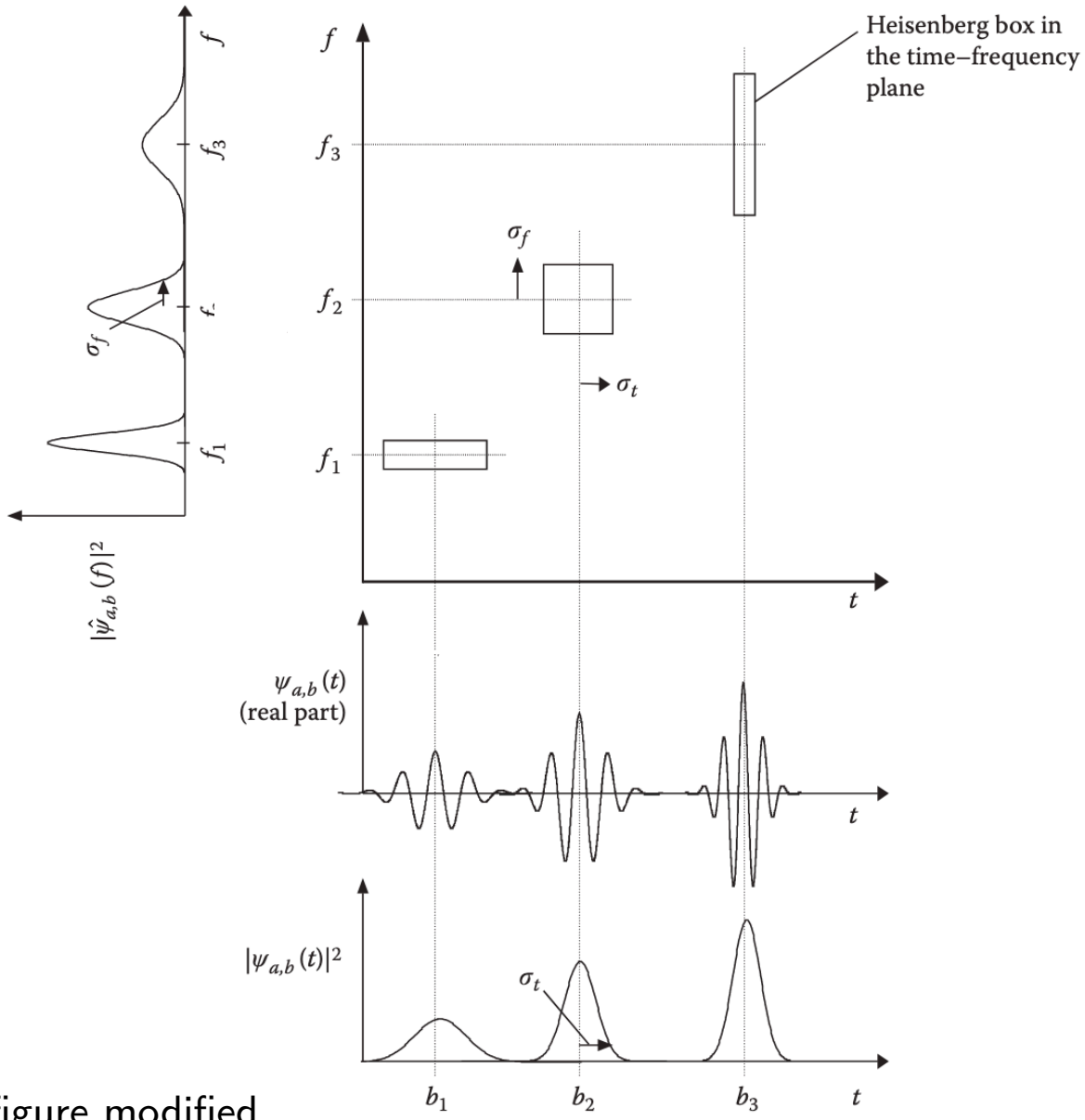
(c) STFT



(d) Wavelets dyadics

Take home message

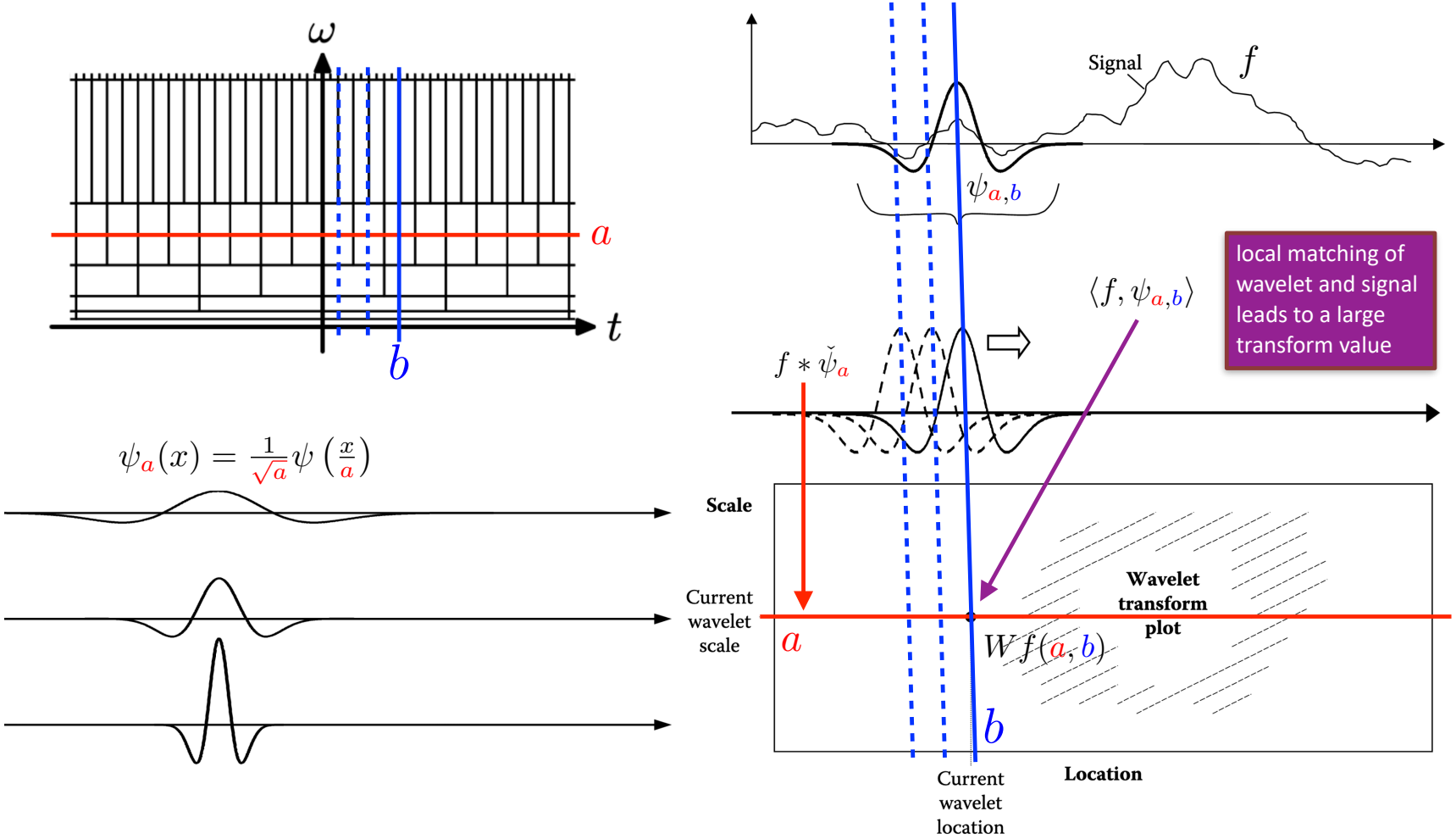
Heisenberg for wavelets



Credits: Paul S Addison's figure modified

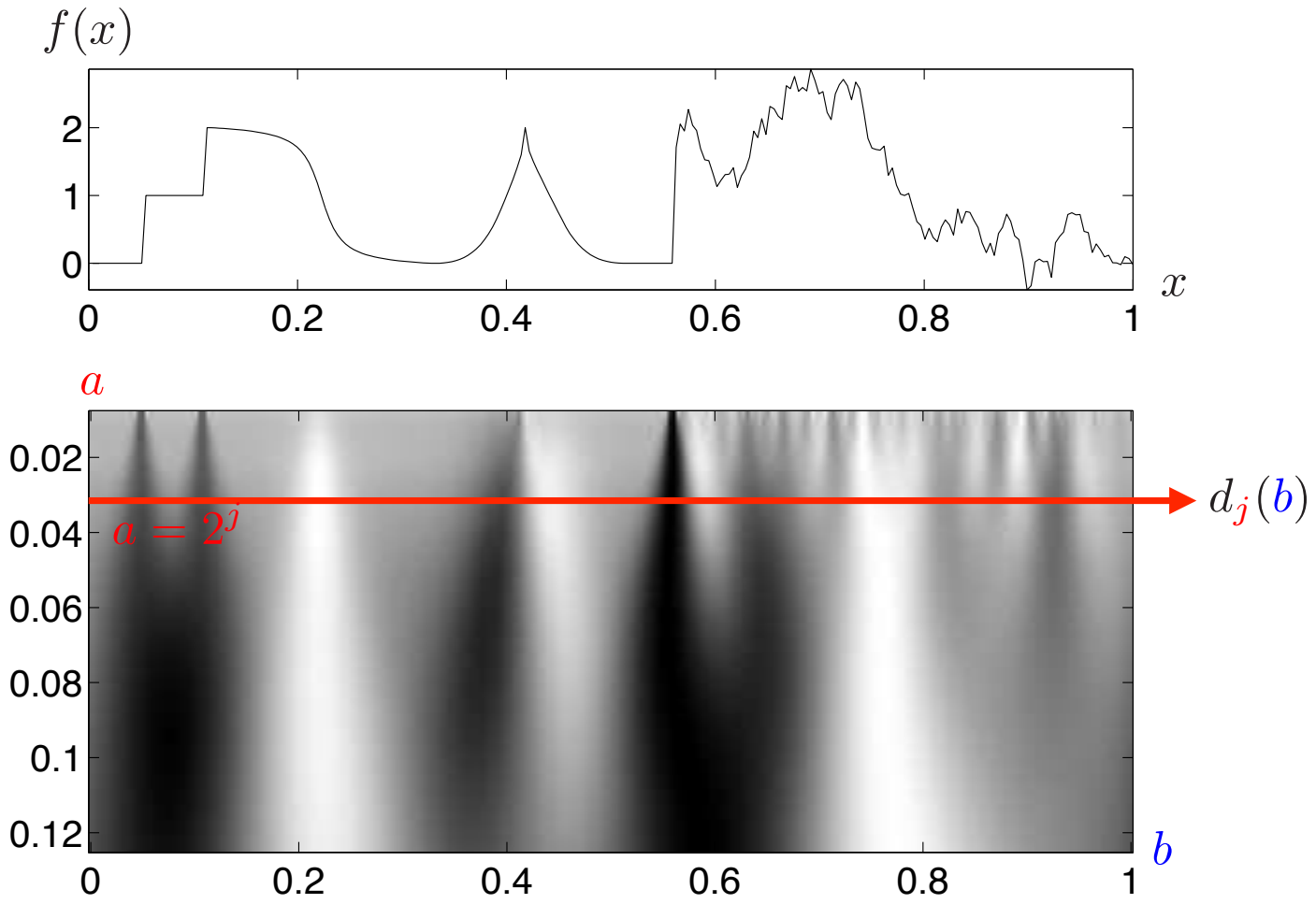
Take home message

Scalogram construction



Credits: Paul S Addison's figure adapted

The Dyadic Wavelet Transform



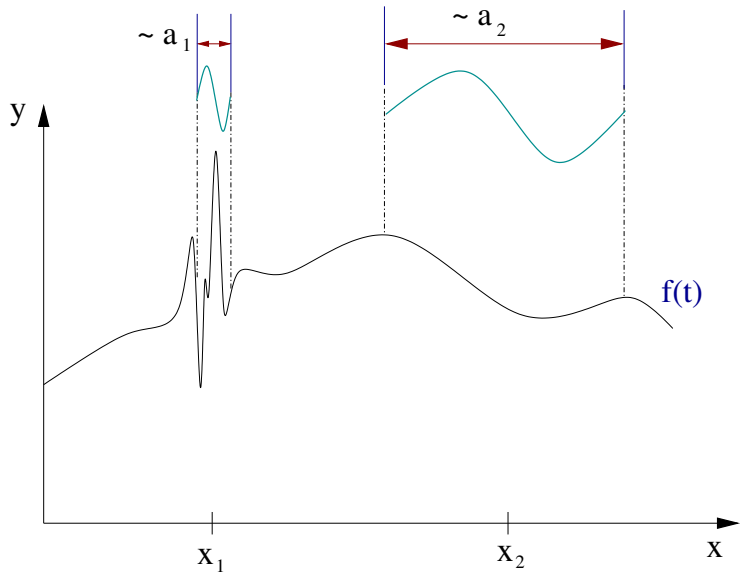
The Dyadic Wavelet Transform

For a fixed a, the CWT is a convolution product:

$$\begin{aligned} Wf(a,b) &= \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} f(x)\, \overline{\psi\left(\frac{x-b}{a}\right)}\, \mathrm{d}x \\ &= (f * \check{\psi}_a)(b) \end{aligned}$$

where we have noted:

$$\psi_a(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x}{a}\right), \quad \check{\psi}_a(x) = \psi_a(-x)$$



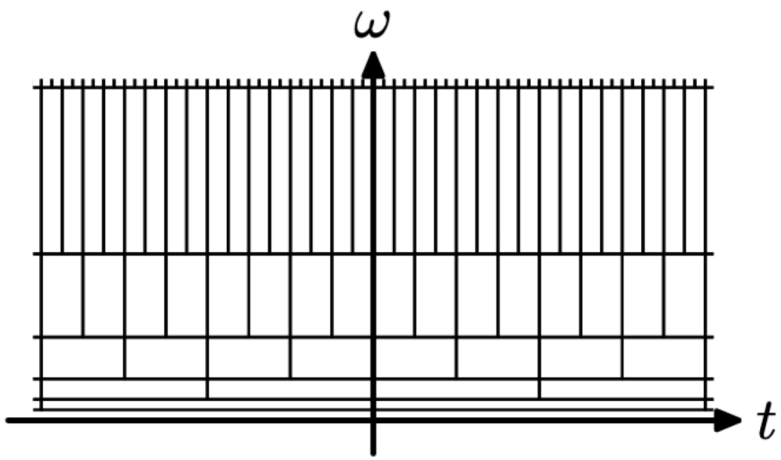
The Dyadic Wavelet Transform

For a fixed $a = 2^j$, the Dyadic Wavelet Transform is a convolution product:

$$\begin{aligned} Wf(2^j, b) &= \frac{1}{\sqrt{2^j}} \int_{-\infty}^{+\infty} f(x)\, \overline{\psi\left(\frac{x-b}{2^j}\right)}\, \mathrm{d}x \\ d_j(b) &= (f * \check{\psi}_j)(b) \end{aligned}$$

where we have noted:

$$\psi_j(x) = \frac{1}{\sqrt{2^j}} \psi\left(\frac{x}{2^j}\right), \quad \check{\psi}_j(x) = \psi_j(-x)$$



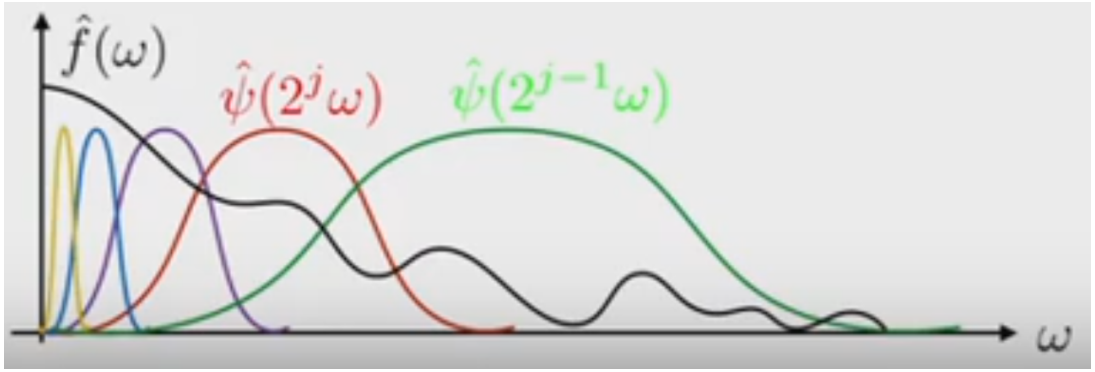
The Dyadic Wavelet Transform

For a fixed $a = 2^j$, the Dyadic Wavelet Transform is a convolution product:

$$\begin{aligned} Wf(2^j, b) &= \frac{1}{\sqrt{2^j}} \int_{-\infty}^{+\infty} f(x)\, \overline{\psi\left(\frac{x-b}{2^j}\right)}\, \mathrm{d}x \\ d_j(b) &= (f * \check{\psi}_j)(b) \end{aligned}$$

whose Fourier transform is

$$\widehat{d}_j(\omega) = \widehat{f}(\omega)\sqrt{2^j}\,\widehat{\psi}^*(2^j\omega)$$



The Dyadic Wavelet Transform

Theorem (Littlewood-Paley, 1930)

If $\sum_j |\widehat{\psi}(2^j\omega)|^2 = 1$ then

$$f(x) = \sum_j 2^{-j} \int Wf(2^j, b)\psi_{2^j,b}(x)\,\mathrm{d}b$$

Proof: Remark that

$$\int Wf(2^j, b)\psi_{2^j,b}(x)\,\mathrm{d}b = d_j * \psi_j(x)$$

then take the Fourier transform

$$\begin{aligned}\sum_j 2^{-j}\widehat{d}_j(\omega)\widehat{\psi}_j(\omega) &= \sum_j 2^{-j}\widehat{f}(\omega)\sqrt{2^j}\widehat{\psi}^*(2^j\omega)\sqrt{2^j}\psi(2^j\omega)\\ &= \widehat{f}(\omega)\underbrace{\sum_j |\widehat{\psi}(2^j\omega)|^2}_{=1} = \widehat{f}(\omega)\end{aligned}$$