

Wavelets and Applications

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M2 MSIAM & Ensimag 3A MMIS

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LABORATOIRE
JEAN KUNTZMANN
MATHÉMATIQUES APPLIQUÉES - INFORMATIQUE

Course Wavelets and Applications – Outline

- 1 Introduction: from Fourier to Wavelets
- 2 The 1D Continuous Wavelet Transform
- 3 Wavelet zoom: a local characterization of functions
- 4 Lab1: 1D Continuous Wavelet Transform
- 5 The 2D Continuous Wavelet Transform
- 6 Wavelet Bases (Haar, multiresolution, orthogonal wavelet bases)
- 7 Lab 2: Fast Wavelet Transform, image compression and denoising
- 8 Fast Wavelet Transform (the 2D case, extensions biorthogonal)
- 9 Approximation in wavelet bases (sparsity, compression, denoising)
- 10 Application 1: The dual-tree complex wavelet transform and the scattering transform for deep learning
- 11 Application 2: Introduction to wavelets on graphs
- 12 Lab 3: Dedicated to the project

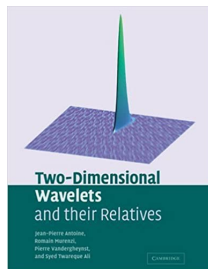
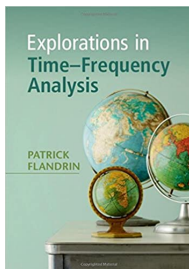
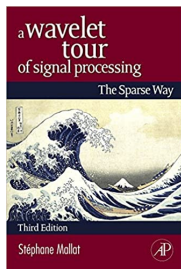
Course evaluation

- ① 2 lab sessions on Matlab using the library WaveLab (results to be sent by email)
- ② Examination. 1 project consisting on:
 - Choosing a research article using wavelets for a given application.
 - Reading and understanding the article, writing a summary of what you expect to implement.
 - Practical work: implementation of the method presented in the article (Matlab or in another language).
 - Writing a report including figures of results.(Ensimag students must form pairs)

Course materials

Books

- **S. Mallat**, *A wavelet tour of signal processing*, Academic press, third edition, 2009.
- **P. Flandrin**, *Explorations in Time-Frequency Analysis*, Cambridge University Press, Cambridge (UK), 2018.
- **J-P. Antoine**, R. Murenzi, P. Vandergheynst and S.T. Ali, *Two-dimensional Wavelets and Their Relatives*, Cambridge University Press, Cambridge (UK), 2004.



Course materials

Links

- **WaveLab** (free Matlab toolbox)

<http://www-stat.stanford.edu/~wavelab/>

- **A numerical tour of Signal/Image Processing** (by Gabriel Peyré)

<http://www.numerical-tours.com/>

- **PyWavelets** (python)

<https://pywavelets.readthedocs.io/>

Introduction

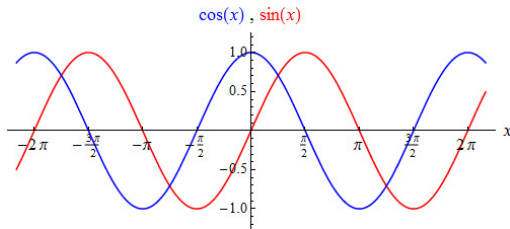
From Fourier to Wavelets

What is a wavelet?

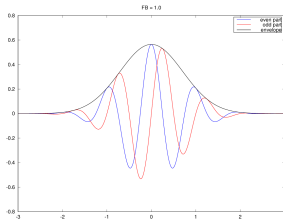
Examples of waves

- Electromagnetic wave
- Radio wave
- Microwave
- Sound wave

Wavelet = "short wave"



sinusoidal waves

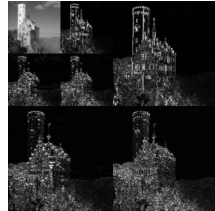
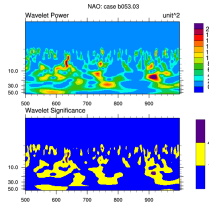
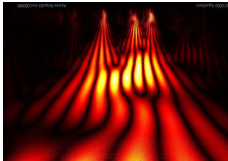


wavelet

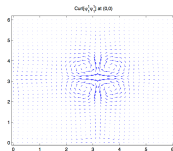
Credits: Valérie Perrier

A success story

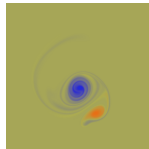
- *Wavelets for Data representation*



- *Wavelets for numerical simulation*



Divergence-free wavelet



Direct Simulation of Turbulence



A success story



WSQ (1993) is the **FBI's Wavelet** Scalar Quantization: it is a national standard for the collecting, encoding, storing, and retrieving digitized fingerprint images.



JPEG 2000 is an image coding system that uses state-of-the-art compression techniques based on **wavelet technology**.



Academy Sci-Tech Award 2013

Awarded to **Theodore Kim, Nils Thuerey, Dr. Markus Gross** and **Doug James** for the invention, publication and dissemination of "**Wavelet Turbulence**" software.

Credits: Valérie Perrier

A success story

- **Abel Prize (2017)** : Yves Meyer, for his pivotal role in the development of the mathematical theory of wavelets.

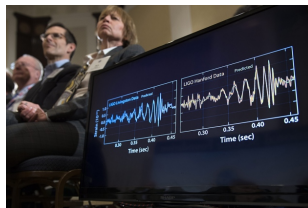


"Wavelet analysis has been applied in a wide variety of arenas as diverse as applied and computational harmonic analysis, data compression, noise reduction, medical imaging, archiving, digital cinema, deconvolution of the Hubble space telescope images, and the recent LIGO detection of gravitational waves created by the collision of two black holes."

[<http://www.abelprize.no/>]

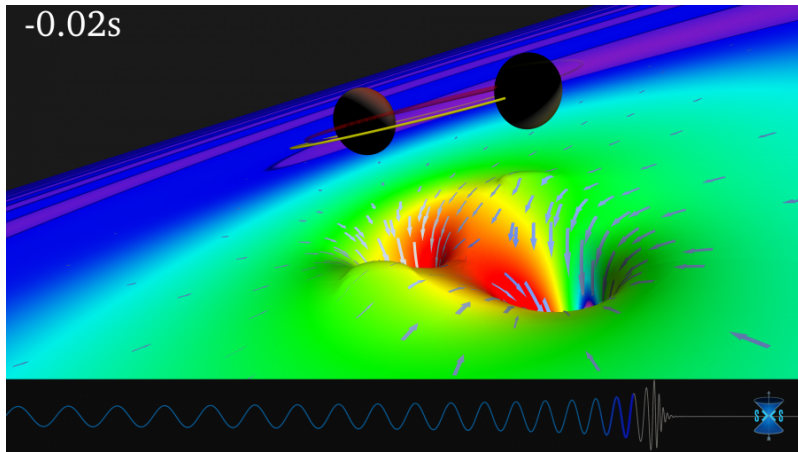
The Abel Lecture (Yves Meyer)

www.youtube.com/watch?v=wxmzHwd3z34



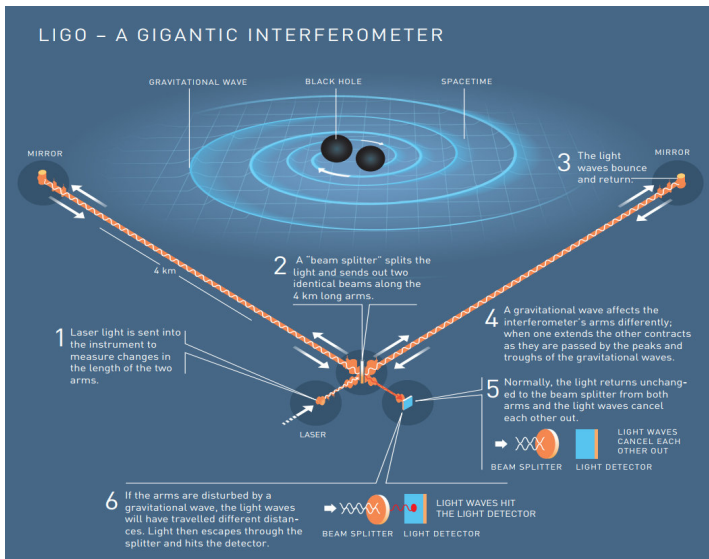
Wavelet theory helped LIGO to detect gravitational waves.

Gravitational waves detection



Credits: LIGO (<http://www.black-holes.org/gw150914>)

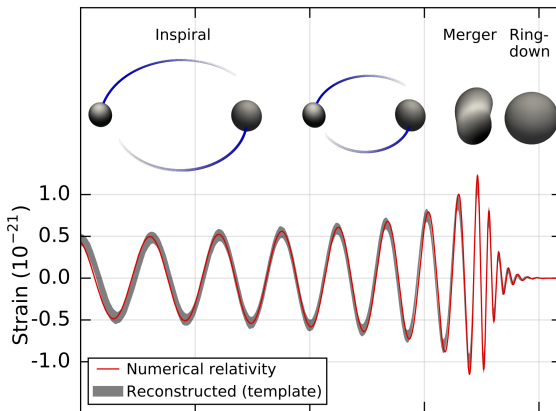
Gravitational waves detection



Credits: LIGO (<http://directory.eoportal.org/web/eoportal/satellite-missions/l/ligo>)

Gravitational waves detection

Analytic wave form derived from Einstein's equations (Thibault Damour et al.)

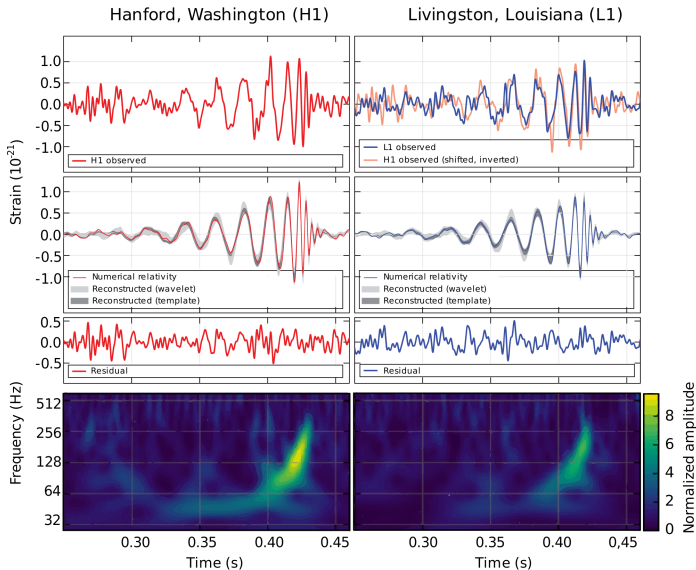


$$s(t) = c|t - t_0|^{-\frac{1}{4}} \cos(\omega|t - t_0|^{\frac{5}{8}} + \phi)$$

with c , ω , ϕ and t_0 are constants (t_0 = time when the stars are merging)

Credits: B.P. Abbott et al., Observation of Gravitational Waves from a Binary Black Hole Merger

Gravitational waves detection



Credits: B.P. Abbott et al., Observation of Gravitational Waves from a Binary Black Hole Merger

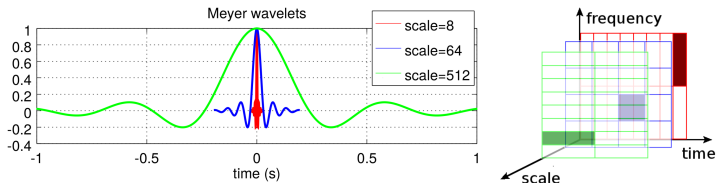
Gravitational waves detection

- ▶ Gravitational waves are perturbation of the space-time metric, predicted by Einstein and steadily sent to the Earth
- ▶ Gravitational waves, produced during the final fraction of second of the merger of two black holes into a single, are **chirps** like a modulated song of a bird, which are **very tenious** (noise/signal $> 10^3$)
- ▶ Pattern detected by the 2 arms of an interferometer, separated by 3000km and 4km long, use laser lights and distant mirrors to be sensitive to deformations smaller than **10^{-19}m** (!)
- ▶ Detecting a short-lived chirp burried inside a very noisy signal cannot be extrated by Fourier analysis (adapted to stationary signals)
- ▶ **Wavelet analysis extends and overcomes Fourier limitation by exploring the time-frequency / time-scale structure of the data.** It explores what is beyond our senses by yielding details that cannot be perceived by our eyes (it acts like a "zoom")

Gravitational waves detection

Klimenko's algorithm: Coherent Wave Burst

- ▶ Projection on bases of functions reasonably localized in time and frequency: Wilson transforms (modification of Gabor transforms)
- ▶ The window can be Meyer scaling function. The signal processing is performed on 7 Wilson bases (and their quadrature bases) each obtained by a dilation of factor 2 of the window
- ▶ Several decompositions at different time scales. Inspiral requires good frequency resolution, merger requires good time resolution: compromise between time frequency and time scale analysis
- ▶ Retain time-frequency pixels that are “phase coherent”. Can detect unexpected sources. Fast and robust algorithm.



Credits: Eric Chassande-Mottin

Gravitational waves detection

Klimenko's algorithm: Coherent Wave Burst

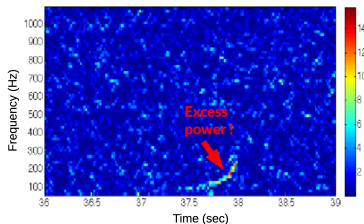
Search for rare transients with low signal to noise ratio. Two ways:

- ▶ **Matched filtering.** Expected signal is known, targeted search signature of binary black-hole merger as predicted by general relativity
- ▶ **Time-frequency excess power.** Expected signal is unknown, search transients appearing in phase in all detectors with no waveform prior (general relativity not needed)



Y. Meyer & S. Klimenko

Credits: Yves Meyer & Eric Chassande-Mottin

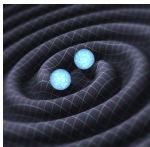


Non-parametric search

Gravitational waves detection

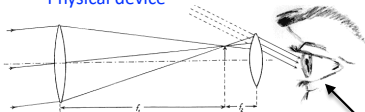


Astronomical objects

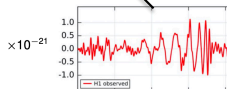
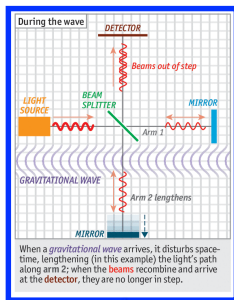


Wavelet analysis
explores beyond
our senses

Physical device

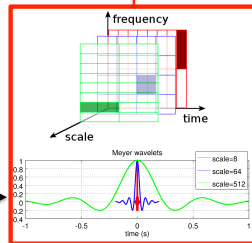


Reconstruction

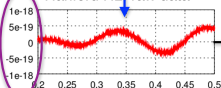


Transient signals
extraction

Klimenko's
algorithm



Hanford H1 raw data



Credits: S. Klimenko et al., arXiv:1511.05999 + The economist

A new window on the universe

“ *It's like Galileo pointing the telescope for the first time at the sky. You're opening your eyes — in this case, our ears — to a new set of signals from the universe that our previous technologies did not allow us to receive, study and learn from.* ”

Vassiliki Kalogera

“ *Up until now, we've been deaf to gravitational waves. What's going to come now is we're going to hear more things, and no doubt we'll hear things that we expected to hear... but we will also hear things that we never expected.* ”

David Reitze

“ *Writing the score while listening to the music, then analyzing and interpreting the score, is, in a sense, what Klimentenko does to detect gravitational waves.* ”

Yves Meyer

From the music of the spheres to the chirp of black holes

A small detour through old cosmology to meet Fourier

Back to the time of Ptolemy...

Who Wants to Be a Millionaire?

A pre-Copernican TV show



Credits: "Qui veut gagner des millions" (<http://www.youtube.com/watch?v=ekmtqODjrSI>)

The first heliocentric model by Aristarchus of Samos

Extract from The Sand Reckoner, Arenarius (Archimedes, c. 230 BC)

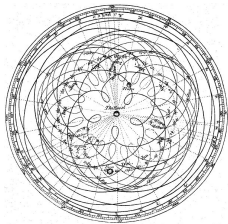
“ You are now aware that the "universe" is the name given by most astronomers to the sphere, the centre of which is the centre of the earth, while its radius is equal to the straight line between the centre of the sun and the centre of the earth. This is the common account as you have heard from astronomers. But **Aristarchus** has brought out a book consisting of certain hypotheses, wherein it appears, as a consequence of the assumptions made, that the universe is many times greater than the "universe" just mentioned. *His hypotheses are that the fixed stars and the sun remain unmoved, that the earth revolves about the sun on the circumference of a circle, the sun lying in the middle of the orbit*, and that the sphere of the fixed stars, situated about the same centre as the sun, is so great that the circle in which he supposes the earth to revolve bears such a proportion to the distance of the fixed stars as the centre of the sphere bears to its surface. ”

Archimedes

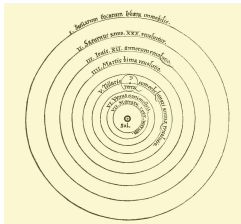
Credits: Sir Thomas Heath, Aristarchus of Samos, the ancient Copernicus (1913)

Three competing models of the solar system

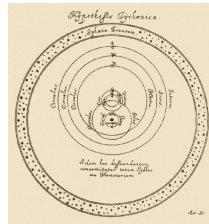
Ptolemy, Copernic and Tycho-Brahe models



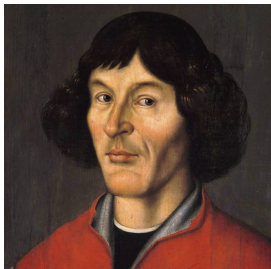
(g) Ptolemy



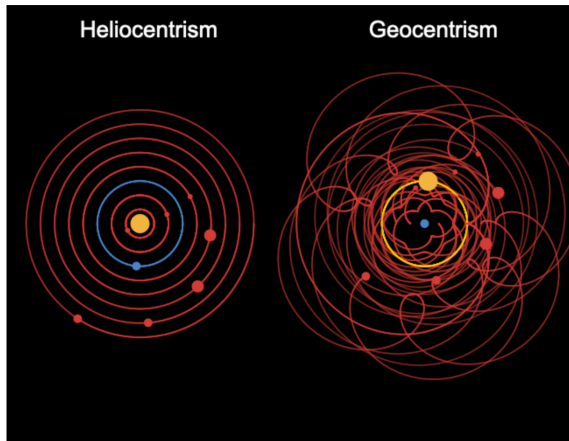
(h) Copernic



(i) Tycho-Brahe



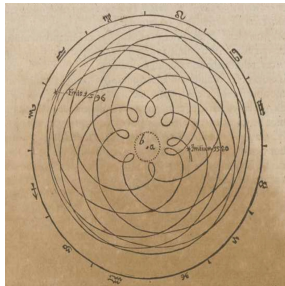
Heliocentrism versus Geocentrism



Remark: not realistic (the Copernican model is much more complex), it just serves to illustrate the relativity of motion.

Credits: Malin Christersson (<http://www.malinc.se/math/trigonometry/geocentrismen.php>)

Mars apparent retrograde motion



Credits: Tunc Tezel & Robert Rynasiewicz

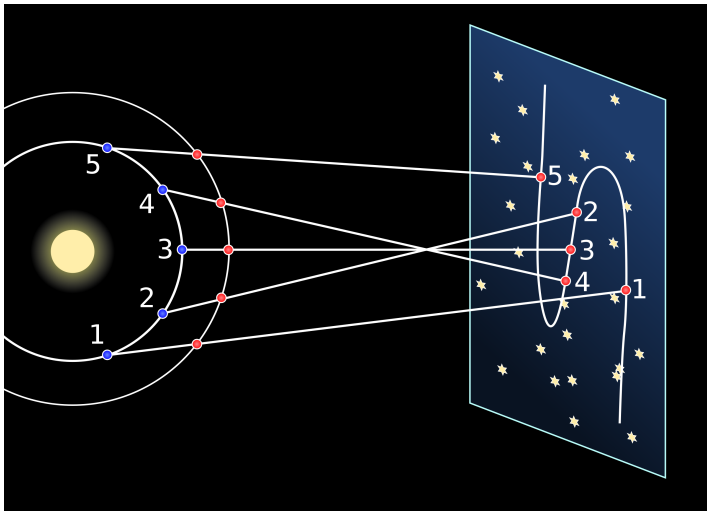
Mars apparent retrograde motion



“ If anyone were to believe that the sun truly moves in the space of a year through the zodiac, which **Ptolemy** and **Tycho Brahe** believed, then it is necessary to concede that the paths of the three superior Planets through ethereal space, composed as they are of several motions, are in reality spirals in the figure of a **pretzel**, in the general fashion that follows.”

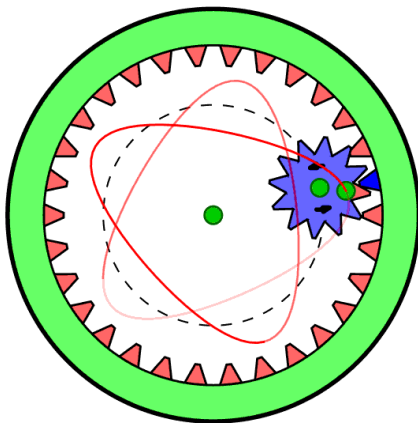
Johannes Kepler

Mars apparent retrograde motion



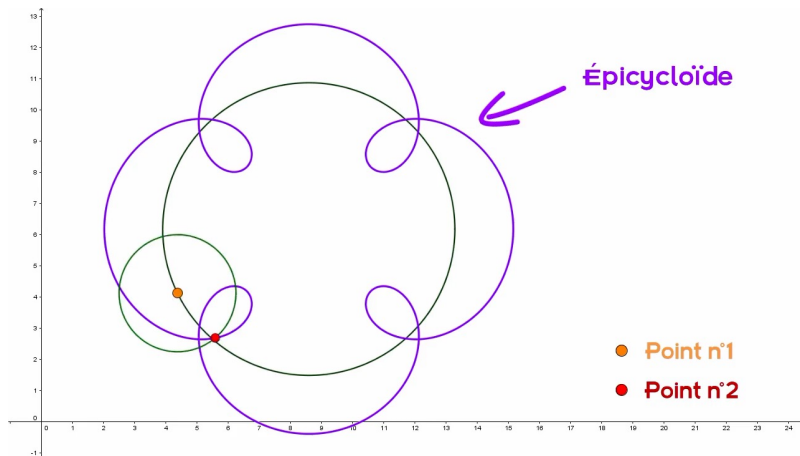
Credits: Wikipedia (https://en.wikipedia.org/wiki/Apparent_retrograde_motion)

Spirograph



Credits: Wikipedia (<http://en.wikipedia.org/wiki/Spirograph>)

Epicycles



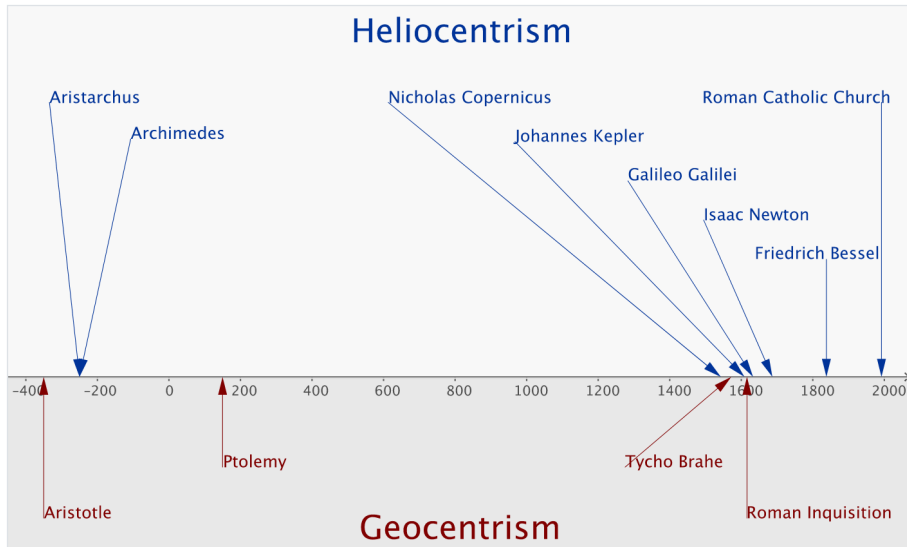
Credits: El Jj (<http://www.youtube.com/watch?v=uazPP0ny3XQ>)

Roots of epicycles: from Hipparque to Kepler

- ▶ **Hipparque's** epicycles theory based on **Apollonius of Perga's** works (at the end of the 3rd century BC) + **Seleucus of Seleucia**
- ▶ **Ptolemy** refined the deferent-and-epicycle concept and introduced the equant as a mechanism for accounting for velocity variations in the motions of the planets
- ▶ **Copernicus** and his contemporaries were therefore using **Ptolemy's** methods and finding them trustworthy well over a thousand years after **Ptolemy's** original work was published
- ▶ In keeping with past practice, **Copernicus** used the deferent/epicycle model in his theory but his epicycles were small and were called *epicyclets*
- ▶ **Copernicus** eliminated **Ptolemy's** somewhat-maligned equant but at a cost of additional epicycles. Various 16th-century books based on **Ptolemy** and **Copernicus** use about equal numbers of epicycles

Credits: Wikipedia (http://en.wikipedia.org/wiki/Deferent_and_epicycle)

Heliocentrism versus Geocentrism



Credits: Malin Christersson (<http://www.malinc.se/math/trigonometry/geocentrismen.php>)

Heliocentrism versus Geocentrism

Why wasn't the heliocentric model capable of replacing the geocentric one?

Heliocentrism	Geocentrism
<ul style="list-style-type: none">++ Explains retrograde motion++ Smaller epicycles used and avoiding the equants= Good accuracy for determining (mostly) planets location= Circular and uniform motions-- Fails to explain the divergence of Mars to observations-- Movements of the Earth around the sun and on its axis contradict some observations (parallax, stars size, stability, ...)-- Philosophical breakthrough: the Earth is not a unique body anymore, which contradicts both Aristotle and biblical arguments	<ul style="list-style-type: none">-- Cannot-- Artificial equants used for differences of velocities observed= Good accuracy for determining (mostly) planets location= Circular and uniform motions-- Fails to explain the divergence of Mars to observations++ Earth is stationary. Tycho Brahe proposed a mixed model called geoheliocentric observing deferents of Mars and the Sun crossing.++ Philosophically compliant with Aristotle's conception of sublunary/aether distinction and with holy scriptures

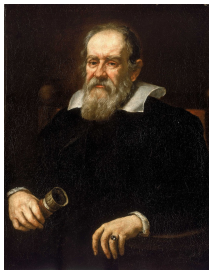
The Paradigm shift: Kepler, Galileo and Newton discoveries

Newton standing on the shoulders of giants

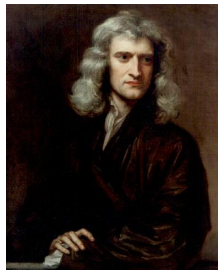
- ▶ The intellectual climate of the time "remained dominated by Aristotelian philosophy and the Ptolemaic astronomy. **At that time there was no reason to accept the Copernican theory, except for its mathematical simplicity.**" **Tycho Brahe's** system ("that the earth is stationary, the sun revolves about the earth, and the other planets revolve about the sun") also directly competed with Copernicus.



Kepler



Galileo



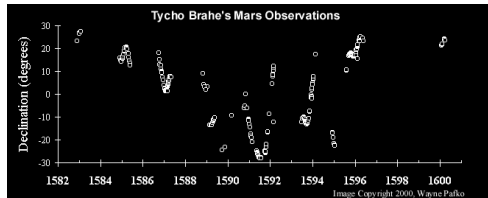
Newton

Credits: Wikipedia (http://en.wikipedia.org/wiki/Nicolaus_Copernicus)

The Paradigm shift: Kepler, Galileo and Newton discoveries

Newton standing on the shoulders of giants

- **Johannes Kepler** developed his laws of planetary motion using measurements made at Tycho's observatory. In *Astronomia nova* (1609), **Kepler** made a diagram of the movement of Mars in relation to Earth if Earth were at the center of its orbit, which shows that Mars' orbit would be completely imperfect and never follow along the same path. To solve the apparent derivation of Mars' orbit from a **perfect circle**, **Kepler** derived both a mathematical definition and, independently, a matching **ellipse** around the Sun to explain the motion of the red planet. **Kepler's laws were born.**

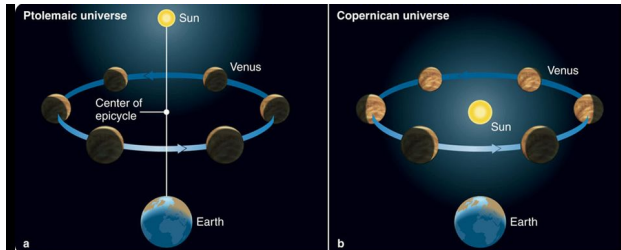


Credits: Wayne Pafko (<http://www.pafko.com/tycho/observe.html>)

The Paradigm shift: Kepler and Galileo discoveries

Newton standing on the shoulders of giants

- **Galileo** was able to look at the night sky with the newly invented telescope. He published his discoveries that **Jupiter is orbited by moons** and that **the Sun rotates** in his Sidereus Nuncius (1610)[93] and Letters on Sunspots (1613), respectively. Around this time, he also announced that **Venus exhibits a full range of phases** (satisfying an argument that had been made against Copernicus). Finally he discovered that the moon presents mountains, valleys and craters which depreciates the Aristotelian conception.



Credits: Pierce Wilcox

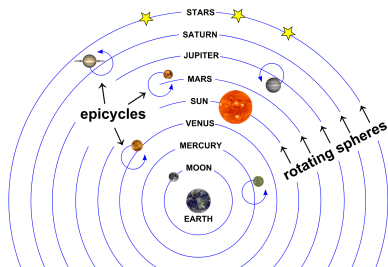
The Paradigm shift: Kepler and Galileo discoveries

Newton standing on the shoulders of giants

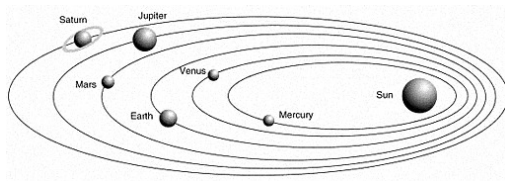
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- ▶ **Galileo** formulated the **principle of inertia** which helped to explain why everything would not fall off the earth if it were in motion.
- ▶ **Isaac Newton** formulated the **universal law of gravitation and the laws of mechanics** in his 1687 Principia, which unified terrestrial and celestial mechanics, was the heliocentric view generally accepted.

Credits: Wikipedia (<http://en.wikipedia.org/wiki/Heliocentrism>)

Why was Ptolemy's system so efficient?



≈



Credits: Joshua Hershey & Universe Today

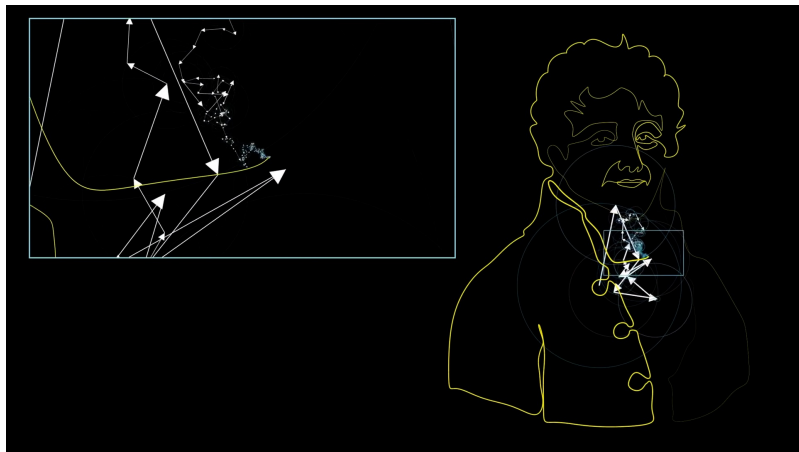
Because of that...



Credits: Carman & Serra (<http://www.youtube.com/watch?v=QVuU2YCwHjw>)

... or more exactly thanks to him: Joseph Fourier

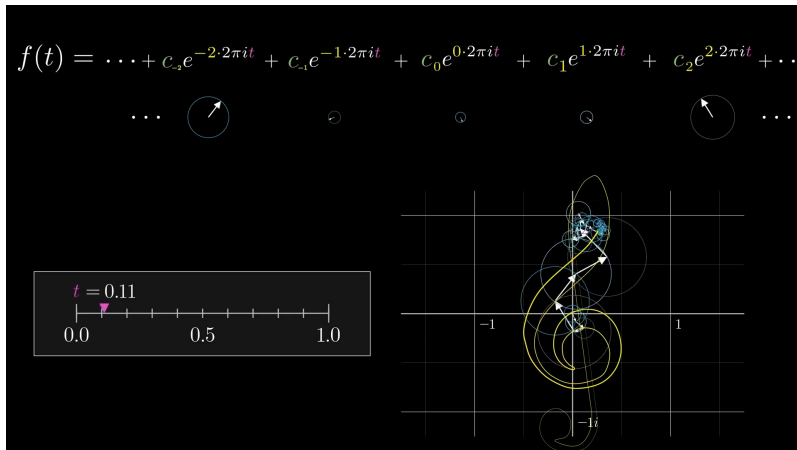
Just like Mr Jourdain speaking prose, astronomers made Fourier series without realizing it



Credits: 3blue1brown (<http://www.youtube.com/watch?v=-qgreAUppwM>)

Fourier series: an intuition behind the decomposition

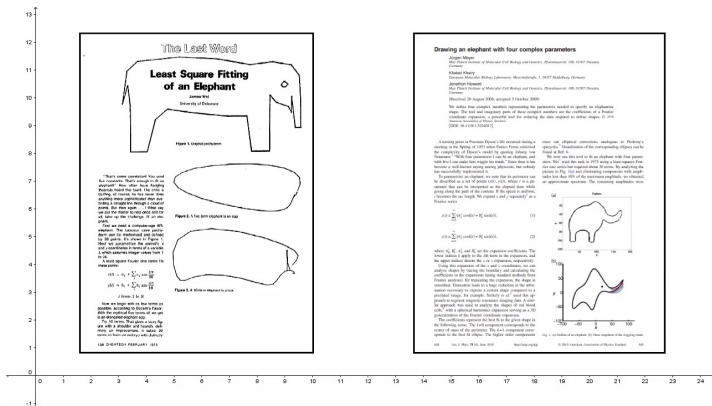
Periodic signals can be decomposed onto the Fourier basis



Credits: 3blue1brown (<http://www.youtube.com/watch?v=r6sGWTMz2k>)

Draw me a (light weight) elephant

The fewer parameters the better



“With four parameters I can fit an elephant, and with five I can make him wiggle his trunk.”

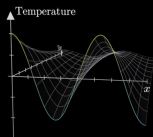
John von Neumann

Credits: El Jj (<http://www.youtube.com/watch?v=uazPP0ny3XQ>)

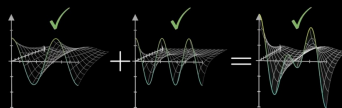
Fourier series: is it only useful for drawing?

How Joseph Fourier solved the heat equation

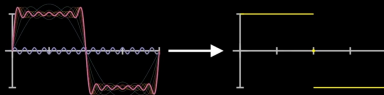
1) Sine = Nice



2) Linearity



3) Fourier series



Credits: 3blue1brown (<http://www.youtube.com/watch?v=r6sGWTCMz2k>)

Fourier series analysis

The **Fourier analysis** decomposes a signal (function) $f(x)$ ($x = \text{times}$) into a sum of sinusoidal functions:

- For a T -periodic function f , with $f \in L^2(0, T)$:

$$f(x) = \sum_{n \in \mathbb{Z}} c_n(f) e^{2i\pi \frac{n}{T} x} \quad (\text{synthesis})$$

where the Fourier coefficients are:

$$c_n(f) = \frac{1}{T} \int_0^T f(x) e^{-2i\pi \frac{n}{T} x} dx \quad (\text{analysis})$$

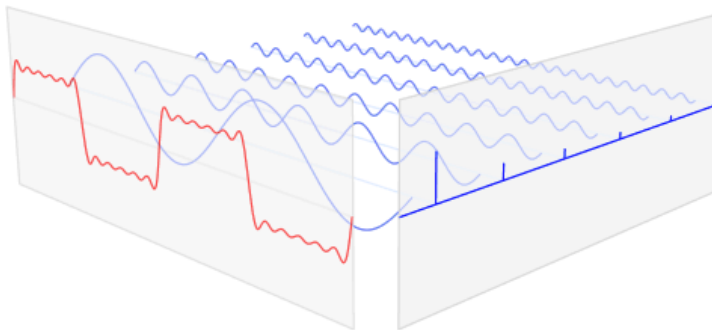
are related to the *frequency* $\frac{n}{T}$ (in Hz).

Parseval equality:

$$\sum_{n \in \mathbb{Z}} |c_n(f)|^2 = \frac{1}{T} \int_0^T |f(x)|^2 dx \quad (\text{energy conservation})$$

Fourier series limitations

Discontinuities require a lot of sinusoids to be described



$$f(x) = \begin{cases} -1 & \text{if } -\pi \leq x < 0 \\ +1 & \text{if } 0 \leq x < \pi \end{cases} = \sum_{n=1}^{+\infty} \frac{4}{\pi(2n-1)} \sin((2n-1)x)$$

Credits: Wikipedia (https://en.wikipedia.org/wiki/Fourier_series)

Fourier transform

- For a function $f \in L^2(\mathbb{R})$:

$$f(x) = \int_{-\infty}^{+\infty} \hat{f}(\nu) e^{2i\pi\nu x} d\nu \quad (\text{synthesis})$$

where the Fourier transform of f is:

$$\hat{f}(\nu) = \int_{-\infty}^{+\infty} f(x) e^{-2i\pi\nu x} dx \quad (\text{analysis})$$

gives information on f for the frequency ν .

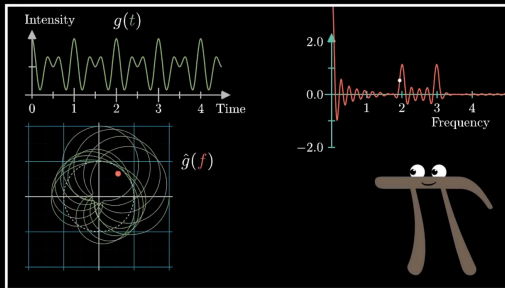
Plancherel-Parseval equality:

$$\int_{-\infty}^{+\infty} |\hat{f}(\nu)|^2 d\nu = \int_{-\infty}^{+\infty} |f(x)|^2 dx \quad (\text{energy conservation})$$

Fourier transform: an intuition behind the transformation

aperiodic signals can also be decomposed onto the continuous dictionary of exponentials

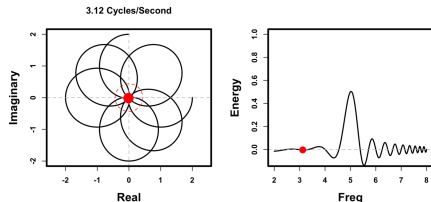
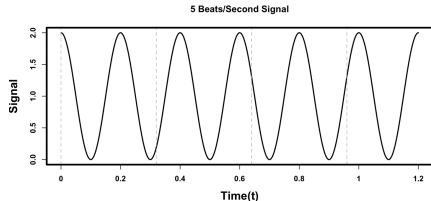
$$\hat{g}(f) = \int_{t_1}^{t_2} g(t) e^{-2\pi i f t} dt$$



Credits: 3blue1brown (<https://www.youtube.com/watch?v=spUNpyF58BY>)

Fourier Transform visualization

Wrap the signal around a circle



$$\hat{g}(f) = \frac{1}{N} \sum_{k=1}^N g(t_k) e^{-2\pi i f t_k}$$

To find **the energy** at a particular frequency, **the signal** is **wrapped around a circle** at the particular frequency and **the points along the path are averaged**.

Fourier transform limitations

Example: two musical notes played at the same time

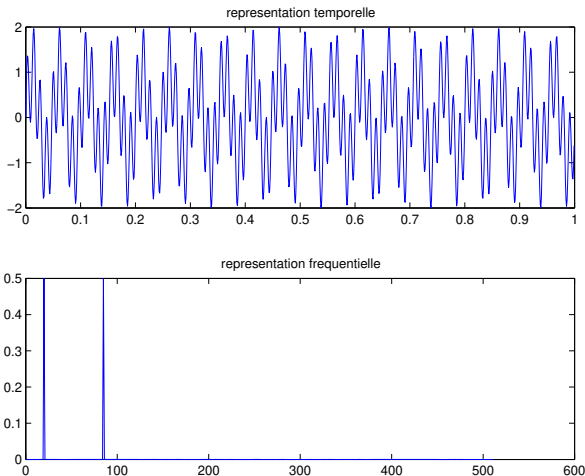
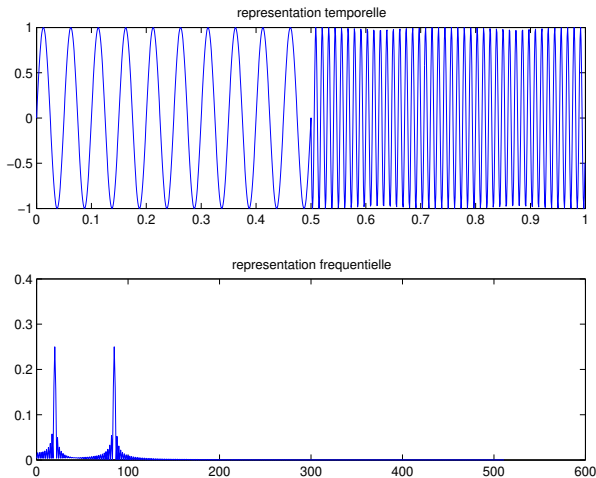


Figure: Signal $f(x) = \sin(40\pi x) + \sin(170\pi x)$ (top), and modulus of its Fourier Transform $\hat{f}(\nu)$ (bottom)

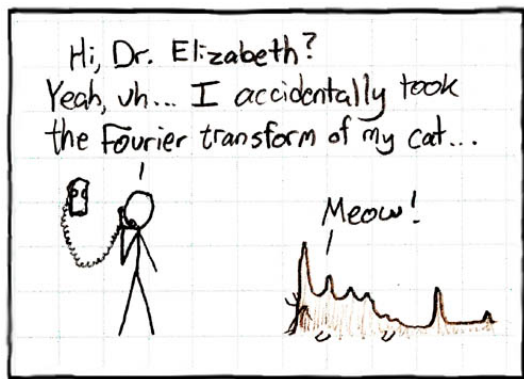
Fourier transform limitations

Example of two musical notes played one after the other

The frequency analysis do not inform on the transient phenomenon in the signal \Rightarrow Loss of temporal localization



Fourier cat transformation



Take home message

To sum up

- Periodic functions (as planets motion along closed orbit) can be approximated by epicycles, that is by Fourier series.
- Fitting data do not necessarily mean that the mechanics behind is understood, and saving the phenomena can lead to a kind of overfitting. What is a good model or a good theory?
- The relativity of motion makes possible to consider different coordinate systems to describe trajectories. Something which is well known by physicists: the choice of the frame of reference can greatly simplify mathematical calculations.
- An appropriate representation of the signal can also reduced the number of parameters needed to encode its information.

⇒ **Toward a sparse representation of signals**

Take home message

To sum up

- Fourier series decomposition allowed Joseph Fourier to solve partial differential equations (heat equation).
 - Extension to the Fourier transform for aperiodic signal also reveals the frequency contents of the signal, but suffer of the same issues:
 - Discontinuities involve a lot of significant coefficients in the decomposition, whose the decrease in amplitudes encodes the *global* regularity of the signal.
 - Losing the temporal localization, the Fourier transform does not allow to capture transient phenomena in the signal.
- ⇒ **Toward a time-frequency representation of signals**