Tight Wavelet Frames on Multislice Graphs

Nora Leonardi, Dimitri Van de Ville

presented by Yusuf Yigit Pilavci

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Outline

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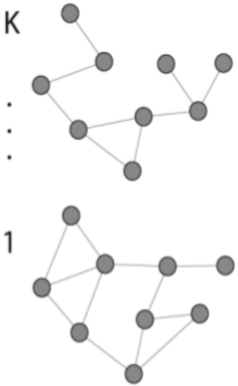

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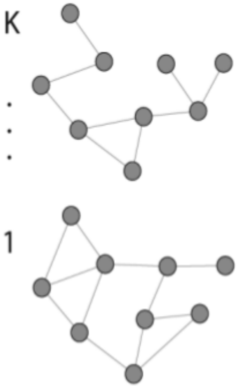
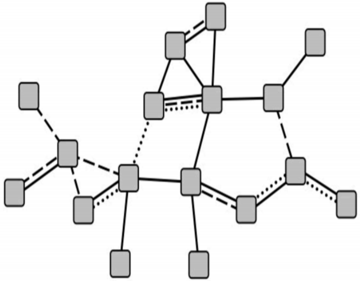
- graphs with varying interactions over time, *i.e.* Dynamic Graphs
- graphs with multiple type of connectivity, *i.e.* Multiplex Graphs





Taken from [SLT10]

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Expectation

Multislice graph wavelets, which can be adapted to the varying graph topology, may give a better suited analysis than the analysis on a single slice.

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A Quick Recap on SGWT

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Variable	Classical	Graph analogy
space variable	x	nodes
Fourier variable	w	λ_i
Fourier basis	e^{-jwx}	$\mathbf{u}_i$
Fourier transform	$\hat{f}(w) = \int\limits_{-\infty}^{\infty} f(x)e^{-jwx}dx$	$\hat{f}(\lambda_i) = \sum\limits_{j=1}^{N} f(j)\mathbf{u}_i(j)$
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- If we put them together, we obtain a set of vectors:

$$F = \{\phi_a\}_{a=1}^{n} \cup \{\psi_{a,s_j}\}_{a=1,j=1}^{n,J}$$

Tight Wavelet Frames

- F forms a frame of $l_2(\mathcal{V})$, if there exists frame bounds $A, B > 0$ such that

$$\forall \mathsf{v} \in l_2(\mathcal{V}), \quad A||f||^2 \leq \sum_{\mathsf{f} \in F} ||\langle \mathsf{f}, \mathsf{v} \rangle||^2 \leq B||f||^2$$

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- Hammond *et.al.*provide:

$$A = \min_{\lambda \in [0, \lambda_N]} G(\lambda)$$
$$B = \max_{\lambda \in [0, \lambda_N]} G(\lambda)$$

where $G(\lambda) = h^2(\lambda) + \sum_{j=1}^{J} g^2(s_j \lambda)$.

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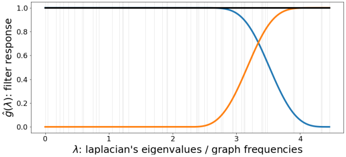
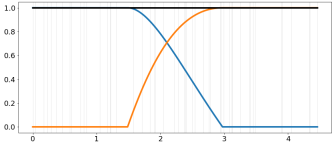
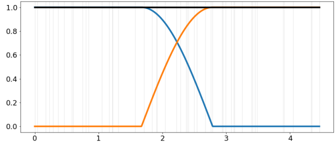
- A Parseval frame is a normalized tight frame with $G(\lambda) = A = B = 1$ for all λ's.
- Various Parseval wavelet frames have been adapted to graphs:



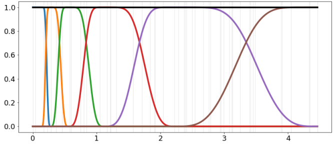
Meyer



Simoncelli



Papadakis



Meyer with more scales

Outline

Tensors and Multislice Graphs

- A tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \dots I_d}$ is an algebraic object that can interpret d dimensional data.

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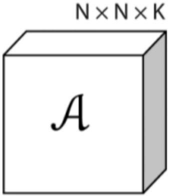
- A tensor $\mathcal{A} \in \mathbb{R}^{I_1 \times I_2 \times \dots I_d}$ is an algebraic object that can interpret d dimensional data.
- A three dimensional tensor $\mathcal{A} \in \mathbb{R}^{N \times N \times K}$ can interpret a multislice graph.



- Each frontal slice $A_{::k}$ is an adjacency matrix.

Tensor Operations

Tensor Operations

- Multiplication between a 3D tensor $\mathcal{A} \in \mathbb{R}^{I \times J \times K}$ and a matrix $\mathrm{B} \in \mathbb{R}^{M \times I}$ is defined as:

$$(\mathcal{A} \times_1 \mathrm{B})_{jkm} = \sum_{i=1}^{I} \mathcal{A}_{ijk} \mathrm{B}_{mi}$$

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- Matricization or Unfolding $\mathrm{A}_{(n)}$ is a reordered concatenation of the slices in the dimension n:

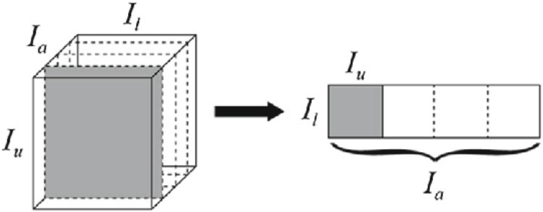


Figure: *Taken from [STK+15]*

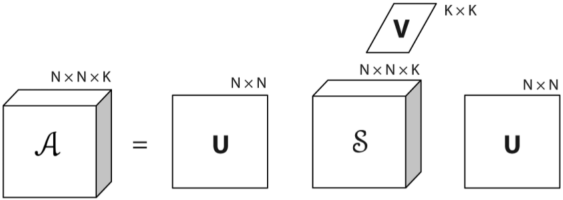
Tensor Analysis

Proposition

A tensor $\mathcal{A} \in \mathbb{R}^{N\times N\times K}$ *with* $\mathrm{A}_{::k} = \mathrm{A}_{::k}^T$ *for all* $k = 1, \dots, K$ *has the the higher order singular value decomposition (HOSVD) decomposition:*

$$\mathcal{A} = \mathcal{S} \times_1 \mathrm{U} \times_2 \mathrm{U} \times_3 \mathrm{V}$$

with $\mathcal{S} \in \mathbb{R}^{N\times N\times K}$ *and* $\mathrm{U} \in \mathbb{R}^{N\times N}$ *and* $\mathrm{V} \in \mathbb{R}^{K\times K}$ *are orthonormal matrices i.e.* $\mathrm{V}^T\mathrm{V} = \mathrm{I}$, $\mathrm{U}^T\mathrm{U} = \mathrm{I}$.



Eigennetworks

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- The first eigennetwork $\mathrm{S}'_{::1}$ turns to be the average of all adjacency matrices.
- Each eigennetwork captures a component of the variation in the edge weights across the networks.

How to compute?

- A quick calculation shows:

$$\mathcal{S}' = \mathcal{A} \times_3 \mathsf{V}^T \iff \mathsf{S}'_{::k} = \sum_{t=1}^{T} \mathsf{V}_{tk} \mathsf{A}_{::t}$$

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- With the unfolding in third dimension $\mathsf{A}_{(3)} = \mathsf{V}\Sigma\mathsf{W}^T$, for $K << N$,one can efficiently compute V by decomposing $\mathsf{A}_{(3)}\mathsf{A}_{(3)}{}^T = \mathsf{V}\Sigma^2\mathsf{V}^T$.

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- Depending on α_t's and $\mathsf{S}_{::t}$, the edge weights associated to *relevant* variation components are emphasized in the new network.
- From the new graph Laplacian $\mathsf{L}' = \mathsf{D}' - \mathsf{A}'$,one has a new SGWT frame on it.

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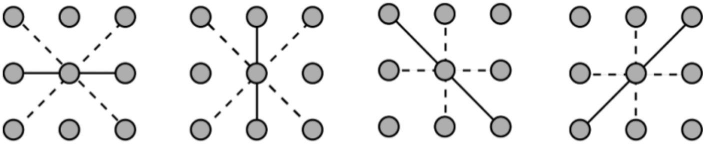

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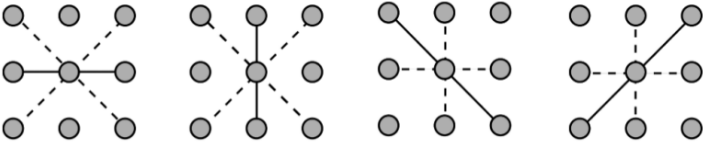

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- The resultant decomposition gives:

$$V = \begin{bmatrix} -0.47 & 0.73 & 0 & 0.59 \\ -0.49 & -0.69 & 0 & 0.54 \\ -0.52 & -0.01 & 0.71 & -0.47 \\ -0.52 & -0.01 & -0.71 & -0.49 \end{bmatrix} \text{ with } S'_{::k} = \sum_{t=1}^{4} V_{tk} A_{::t} \qquad (1)$$

Experiments

- A network set is created from these eigennetworks and a localized filter illustrated on them:

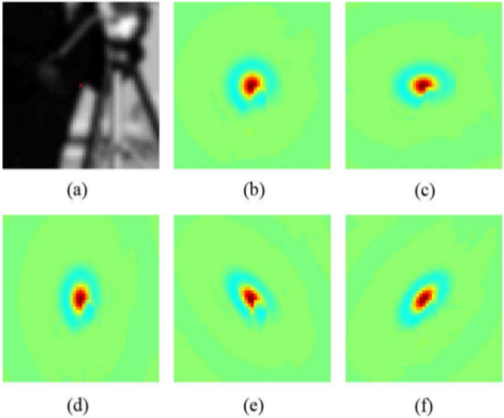

Figure: *(a)filter location,(b)* $A'_1 = -0.5S'_{::1}$, *(c)*$A'_2 = -0.5S'_{::1} + 0.7S'_{::2}$,*(d)*$A'_3 = -0.5S'_{::1} - 0.7S'_{::2}$,*(e)*$A'_4 = -0.5S'_{::1} + 0.8S'_{::3}$,*(f)*$A'_5 = -0.5S'_{::1} - 0.8S'_{::3}$

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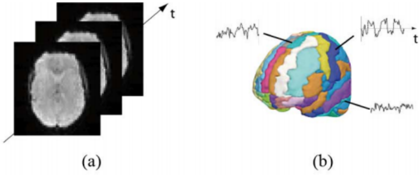

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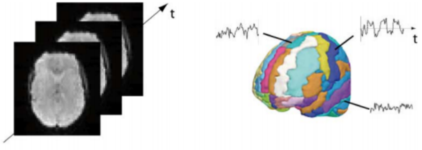
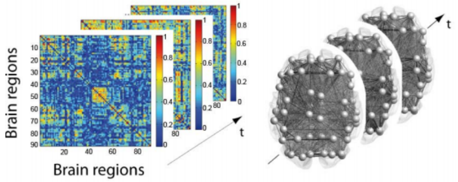


(a) (b)

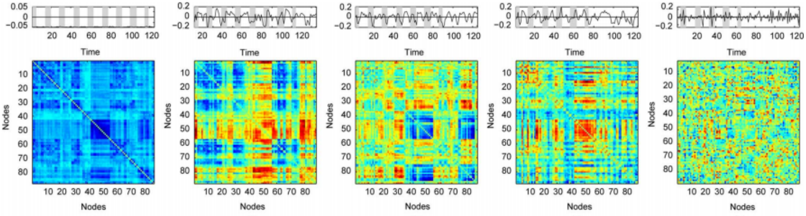
- By a sliding window approach, the correlations of different regions are computed and used as edge weights.



(c) (d)

Experiments:Dynamic Brain Graph

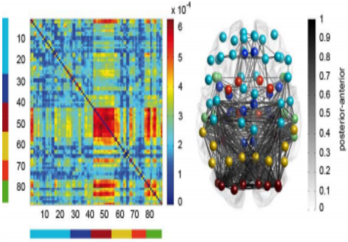
- The eigennetworks $S'_{::k}$'s and v_k's:

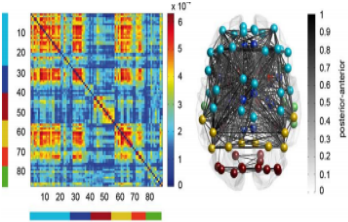


(a) (b) (c) (d) (e)

Experiments: Dynamic Brain Graph

- Two adjacency matrices $A'_1 = -0.1S'_1 + 0.2S'_{::2}$ and $A'_2 = -0.1S'_1 - 0.3S'_{::2}$ are generated:





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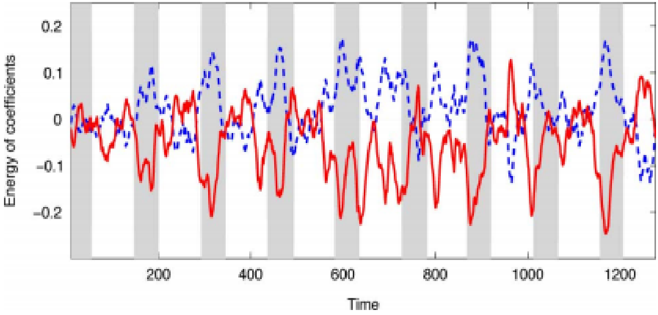
- They are applied to the regional activity signal.
- The energy of scaling and wavelet coefficients are computed in both frame.
- Finally, the difference is plotted:



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- This extension allows us to capture the variation across the graphs.
- It can be used for different GSP tools.

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