Graph Wavelets for Multiscale Community Mining

Nicolas Tremblay and Pierre Borgnat

Presented by Lorenzo Dall'Amico

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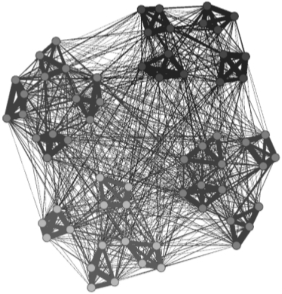
Outline

Table of Contents

Introduction

- What
- Why
- How

What: community mining



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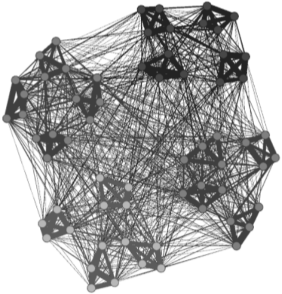

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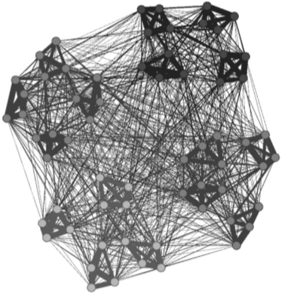

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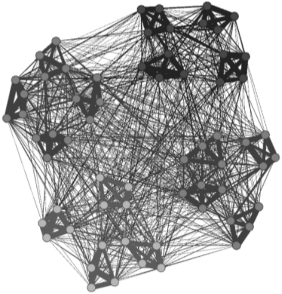

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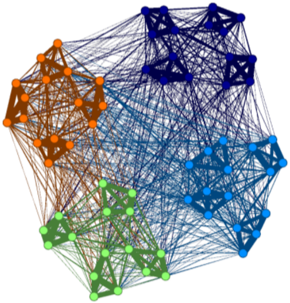

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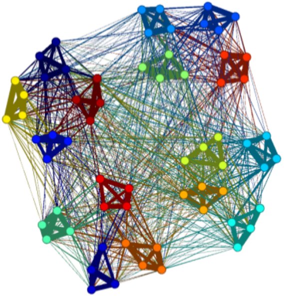

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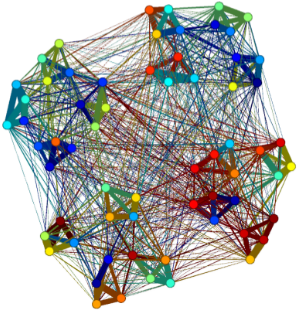

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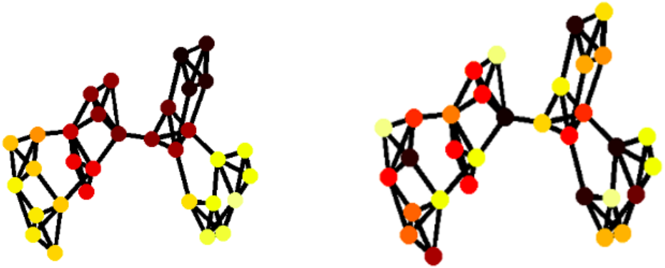
Need of a multiscale approach to community mining

How: graph wavelets



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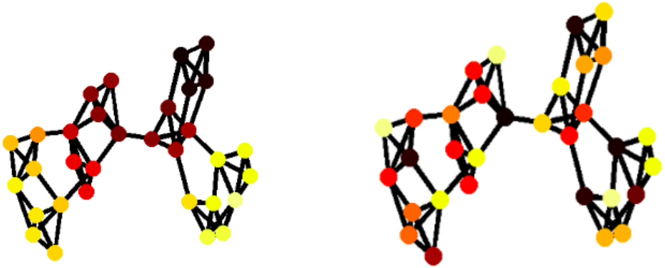

- Centered at a node and spreads over the graph at a given scale
- How the the nodes sees the graph at that scale

Table of Contents

- Spectral clustering
- Graph Fourier transform

Spectral clustering

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- λ_2 Fiedler eigenvalue: community reconstruction

Spectral clustering

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Normalized Laplacian matrix

$$\boldsymbol{L}^{\mathrm{sym}} = \boldsymbol{I_n} - \boldsymbol{D}^{-1/2}\boldsymbol{A}\boldsymbol{D}^{-1/2}$$

Properties

- $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$
- $\forall\, \boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x}^T L^{\mathrm{sym}} \boldsymbol{x} = \frac{1}{2} \sum_{i,j} A_{ij} \left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}} \right)^2$
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Fourier transform (recall)

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Graph Fourier transform

$$\boldsymbol{U} = (\boldsymbol{u}_1|\boldsymbol{u}_2|\dots|\boldsymbol{u}_n) \in \mathbb{R}^{n\times n} \Longrightarrow \hat{\boldsymbol{f}} = \boldsymbol{U}^T\boldsymbol{f}$$

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- Adapt the filter to community mining

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thus the wavelet coefficient of node a at scale s

$$W_f(s,a) = \psi_{\boldsymbol{s},\boldsymbol{a}}^T\boldsymbol{f}$$

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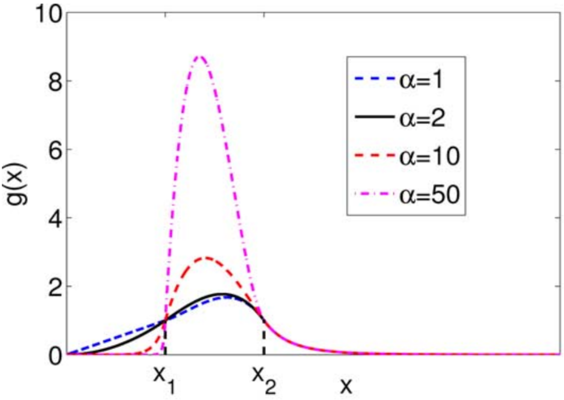
$$W_f(s,a) = \psi_{\boldsymbol{s,a}}^{\boldsymbol{T}}\boldsymbol{f}$$

Project the signal on the wavelet

Adapt the filter to community mining

$$g(x;\alpha,\beta,x_1,x_2)=\begin{cases}\left(\frac{x}{x_1}\right)^{\alpha} & \text{for } x\le x_1\\ p(x) & x_1\le x\le x_2\\ \left(\frac{x_2}{x}\right)^{\beta} & \text{for } x\ge x_2\end{cases}$$



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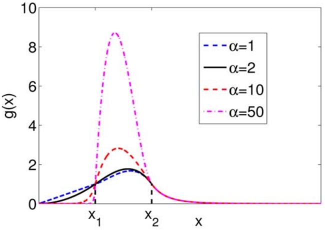

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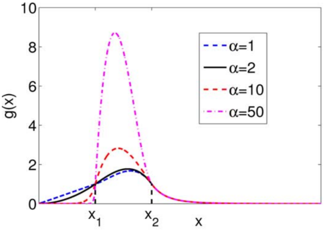

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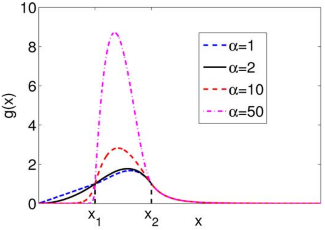

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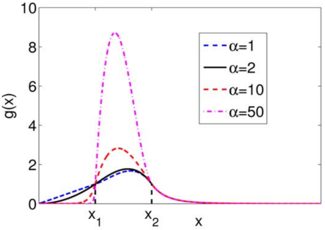

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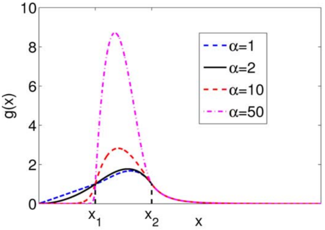

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Fixing $x_1 = 1$, we finally obtain

$$s_{\min} = \frac{1}{\lambda_2}, \quad x_2 = \frac{1}{\lambda_2}, \quad s_{\max} = \frac{1}{\lambda_2^2}$$

Table of Contents

- Identifying the communities
- Fast community mining
- What are the most relevant scales?
- How relevant are the most relevant scales?

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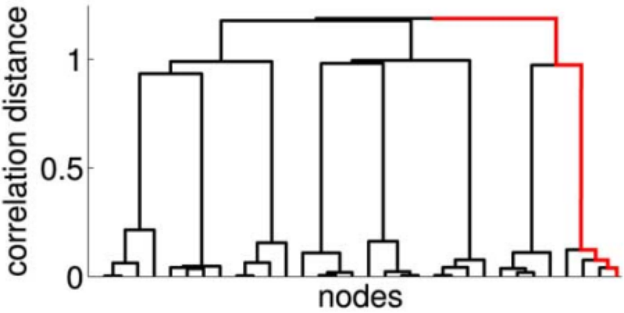
We obtain a dendogram



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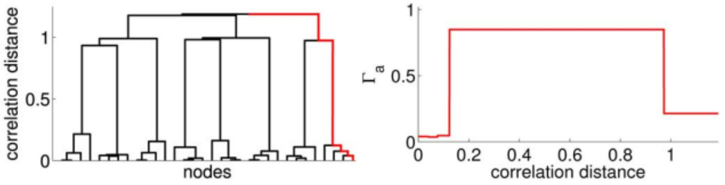
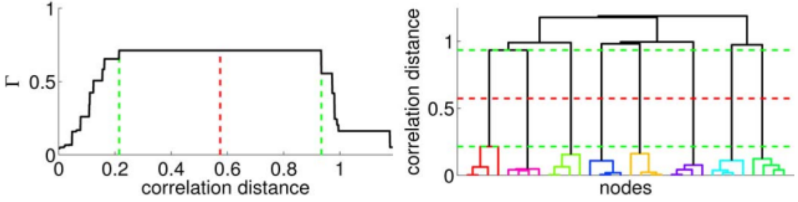


For each node a create the function Γ_a (why is decreasing after one? How is it defined?)



Average Γ_a over all a and obtain Γ. Cut at the maximum of Γ. Repeat for all s and obtain $\{\mathcal{P}_s\}$.

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- For η sufficiently large $\hat{C}_{ab,\eta} \to 1 - D_s(a,b)$
- With Chebishev polynomial approximation, $\boldsymbol{f}_{s,a}$ can be computed efficiently.

$$UG_s U^T \boldsymbol{r} = U \left(\sum_{k=1}^{p} \alpha_k \Lambda_k^k \right) U^T \boldsymbol{r} = \sum_{k=1}^{p} \alpha_k L^k \boldsymbol{r}$$

What are the most relevant scales?

Consider $J(=20)$ sets of η random signals. Obtain the set of partitions $\{\mathcal{P}_s^j\}_{j=1,\dots,J}$.

$$\gamma(s) = \frac{2}{J(J-1)} \sum_{i \neq j} \texttt{ari}(P_s^i, P_s^j)$$

The closer $\gamma(s)$ is to one, the more stable is the partition.

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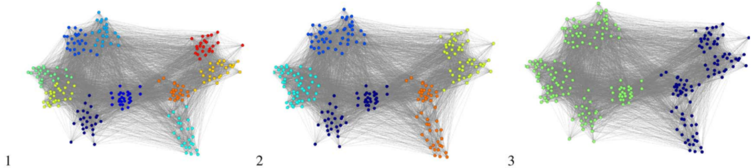
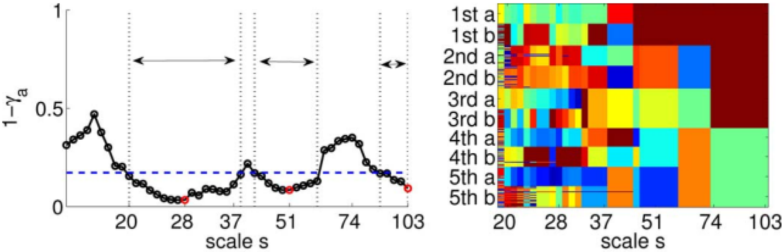
- For each, compute $\gamma_0(s)$ and obtain the empirical distribution
- Reject the configurations with $\gamma(s)$ to close to $\gamma_0(s)$

Table of Contents

An example

A Graph of Social Interactions Between Children in a Primary School





Reference

- Tremblay, Nicolas, and Pierre Borgnat. "Graph wavelets for multiscale community mining." IEEE Transactions on Signal Processing 62.20 (2014): 5227-5239.

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THANKS