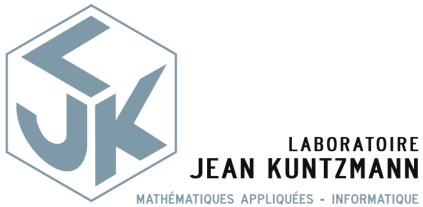
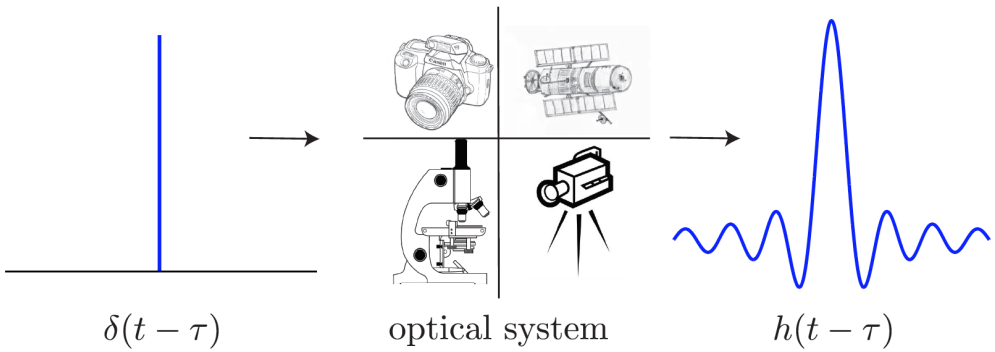


Une approche convexe de la super-résolution et de la régularisation de lignes 2D dans les images

Kévin Polisano, Laurent Condat, Marianne Clausel, Valérie Perrier

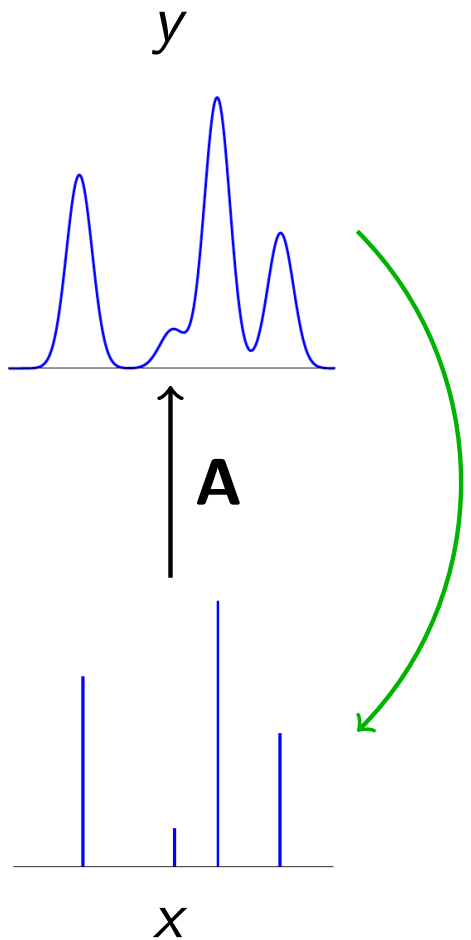
Université Grenoble Alpes, Laboratoire Jean Kuntzmann

Super-resolution of 1-D impulses



$$x(t) = \sum_{k=1}^{K} c_k \delta_{t_k}, \quad c_k \geq 0, \quad t_k \geq 0$$

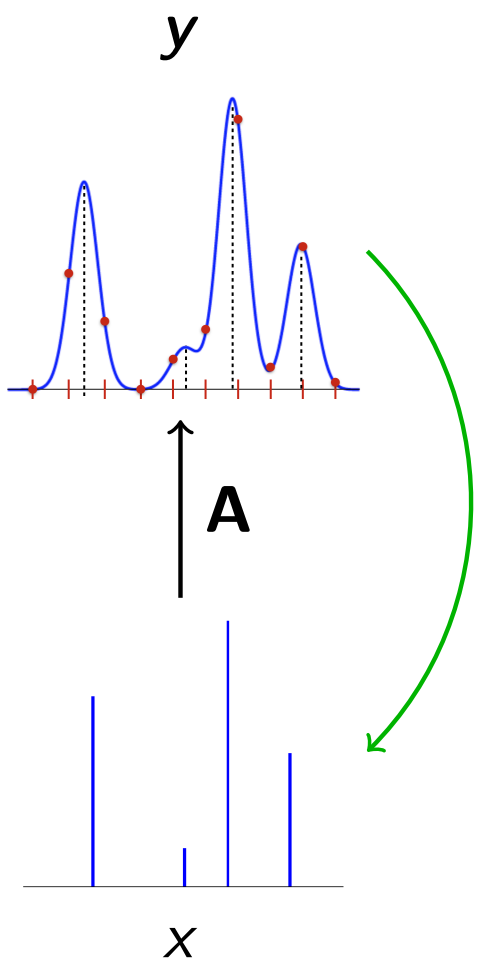
$$y(t) = \sum_{k=1}^{K} c_k h(t - t_k)$$



Sparse ℓ_0 deconvolution on a grid

$$\min_{\boldsymbol{c}\in\mathbb{R}^K} \frac{1}{2}\|\boldsymbol{y}-\mathbf{A}\boldsymbol{c}\|_2^2 + \lambda\|\boldsymbol{c}\|_0$$

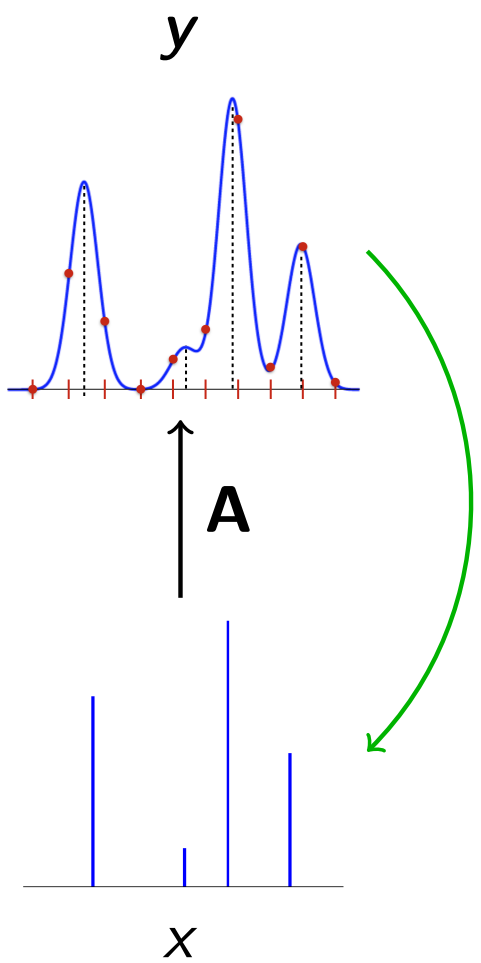
$$\boldsymbol{y} = y(\tau_i), \quad \tau_i = i\Delta/N \longrightarrow \tilde{x}_i$$



Sparse ℓ_1 deconvolution on a grid (LASSO)

$$\min_{\boldsymbol{c}\in\mathbb{R}^K} \frac{1}{2}\|\boldsymbol{y}-\mathbf{A}\boldsymbol{c}\|_2^2 + \lambda\|\boldsymbol{c}\|_1$$

$$\boldsymbol{y} = y(\tau_i), \quad \tau_i = i\Delta/N \longrightarrow \tilde{x}_i$$



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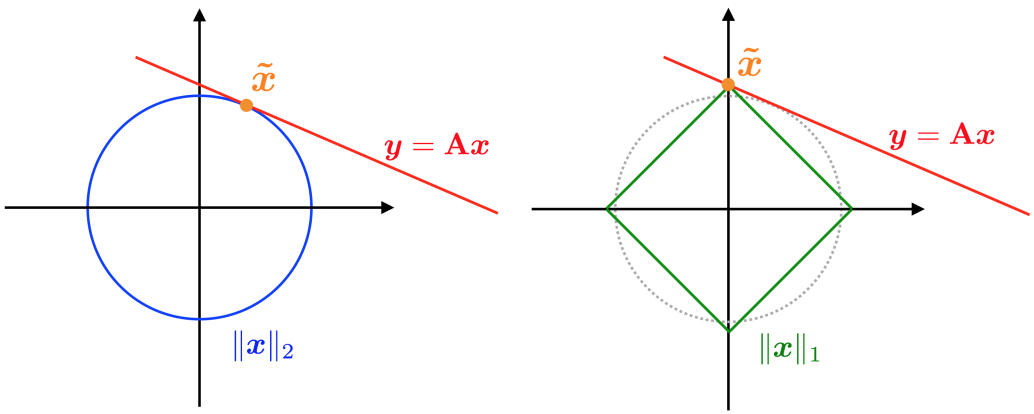
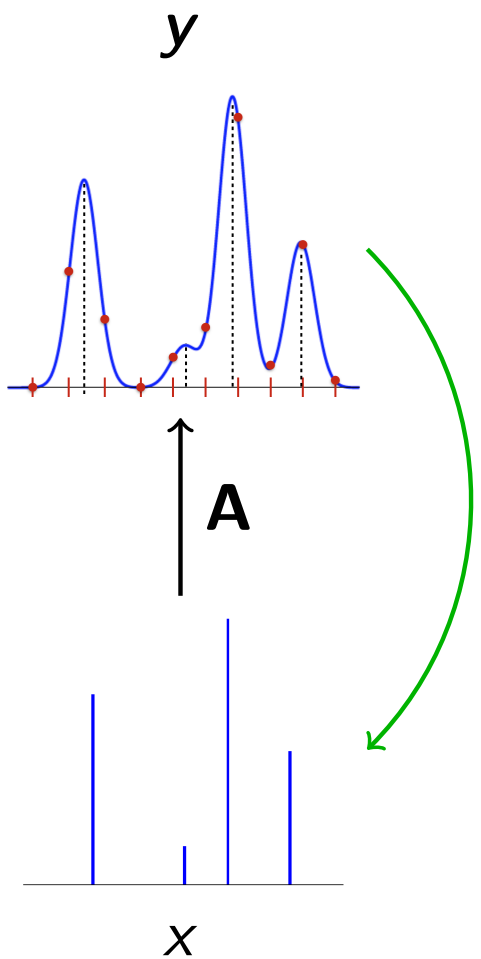
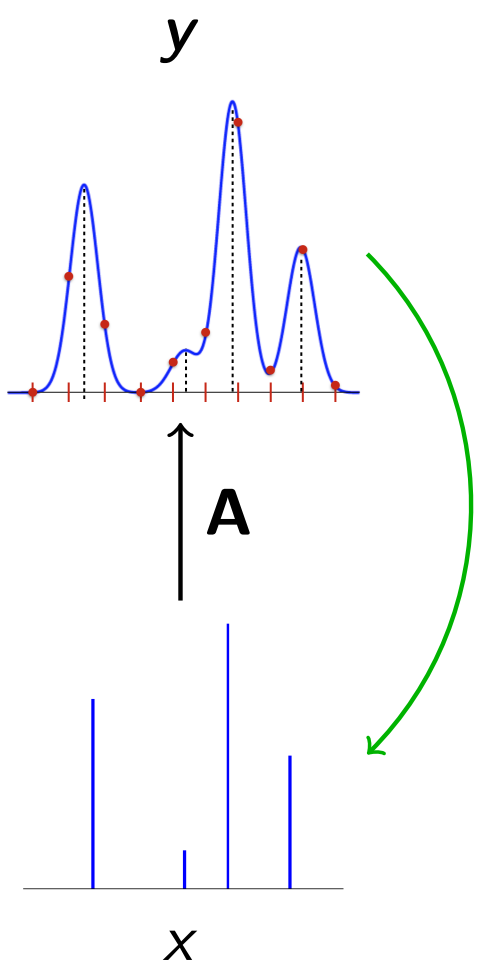




Super-resolution of 1-D impulses off-the-grid

$$x = \sum_{k=1}^{K} c_k \delta_{t_k}, \quad c_k \geq 0, \quad t_k \geq 0$$

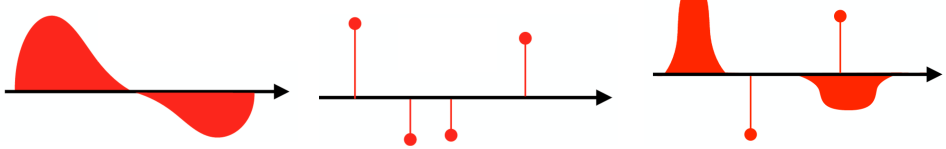


Minimization (convex regularization)

$$\arg\min_{\mu} \frac{1}{2} \|y - \mathbf{A}\mu\|^2 + \lambda \|\mu\|_{\mathrm{TV}}$$

Reference : (Candès, Fernandez-Granda, 2012)

$$\mathrm{d}\mu(t) = f(t)\,\mathrm{d}t$$



$$\|\mu\|_{\mathrm{TV}} = \int |f| \qquad \|x\|_{\mathrm{TV}} = \|c\|_1$$

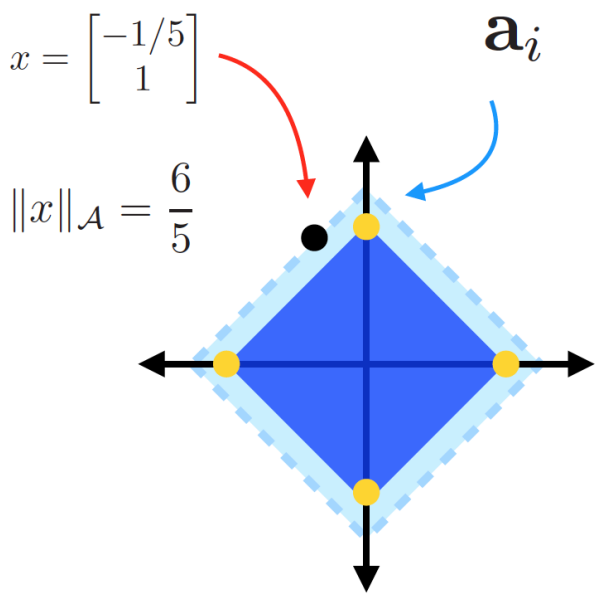
Paradigm of the atomic decomposition

$$\mathbf{x} = \sum_{k=1}^{K} c_k \mathbf{a}_i, \quad c_i \geq 0, \quad \mathbf{a}_i \in \mathcal{A}$$

Atomic norm

$$\|\mathbf{x}\|_{\mathcal{A}} = \inf \{t > 0 : \mathbf{x} \in t\mathrm{conv}(\mathcal{A})\}$$
$$= \inf \left\{ \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} : \mathbf{x} = \sum_{\mathbf{a} \in \mathcal{A}} c_{\mathbf{a}} \mathbf{a} \right\}$$

(Chandrasekaran et al., 2010)



$$\mathcal{A} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$$

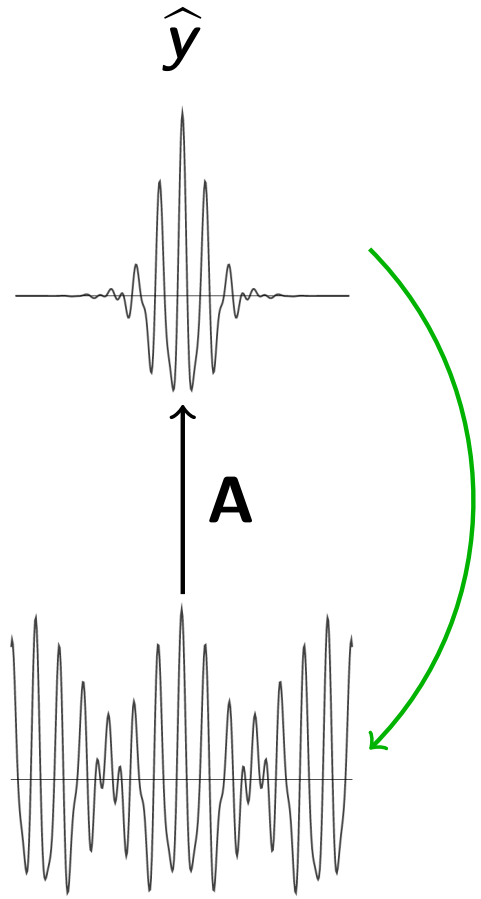
$$\|\mathbf{x}\|_{\mathcal{A}} = \|\mathbf{x}\|_1$$

Super-resolution of 1-D impulses off-the-grid

$$\widehat{\boldsymbol{x}} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_k), \quad c_k \geq 0, \quad \boldsymbol{a}(f_k) \in \mathcal{A}$$

$$\mathcal{A} = \left\{ \boldsymbol{a}(f) \in \mathbb{C}^N \right\}, \quad [\boldsymbol{a}(f)]_i = \mathrm{e}^{\mathrm{j}2\pi f i}$$

$$\|\widehat{\boldsymbol{x}}\|_{\mathcal{A}} = \inf \left\{ \sum_{\boldsymbol{a} \in \mathcal{A}} c_{\boldsymbol{a}} : \widehat{\boldsymbol{x}} = \sum_{\boldsymbol{a} \in \mathcal{A}} c_{\boldsymbol{a}} \boldsymbol{a} \right\}$$



Minimization (convex regularization)

$$\arg\min_{\widehat{\boldsymbol{x}}} \frac{1}{2} \|\widehat{\boldsymbol{y}} - \mathbf{A}\widehat{\boldsymbol{x}}\|^2 + \lambda \|\widehat{\boldsymbol{x}}\|_{\mathcal{A}}$$

Reference : (Tang, Bhaskar, Recht et al., 2013)

$$(\mathcal{F}x)(\omega) = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi f_k \omega}$$

Super-resolution of 2-D impulses off-the-grid

$$\widehat{\mathbf{x}} = \sum_{k=1}^{K} c_k \mathbf{a}(\boldsymbol{f}_k), \quad \boldsymbol{f}_k = (f_{k1}, f_{k2}) \in \mathbb{R}^2$$

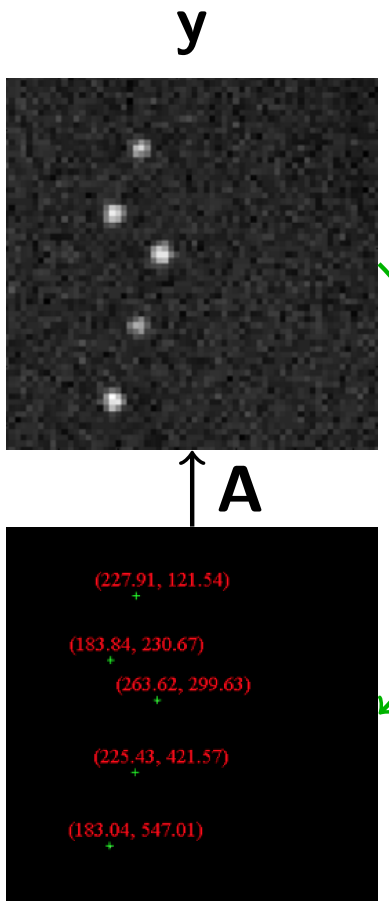
$$\mathcal{A}_{2\mathrm{D}} = \left\{ \mathbf{a}(\boldsymbol{f}) \in \mathbb{C}^{N \times N} \right\}, \ \mathbf{a}(\boldsymbol{f}_k) = \boldsymbol{a}(f_{k1}) \otimes \boldsymbol{a}(f_{k2})$$

$$\|\widehat{\mathbf{x}}\|_{\mathcal{A}_{2\mathrm{D}}} = \inf \left\{ \sum_{\mathbf{a} \in \mathcal{A}_{2\mathrm{D}}} c_{\mathbf{a}} : \widehat{\mathbf{x}} = \sum_{\mathbf{a} \in \mathcal{A}_{2\mathrm{D}}} c_{\mathbf{a}} \mathbf{a} \right\}$$

Minimization (convex regularization)

$$\arg\min_{\widehat{\mathbf{x}}} \frac{1}{2} \|\widehat{\mathbf{y}} - \mathbf{A}\widehat{\mathbf{x}}\|^2 + \lambda \|\widehat{\mathbf{x}}\|_{\mathcal{A}_{2\mathrm{D}}}$$

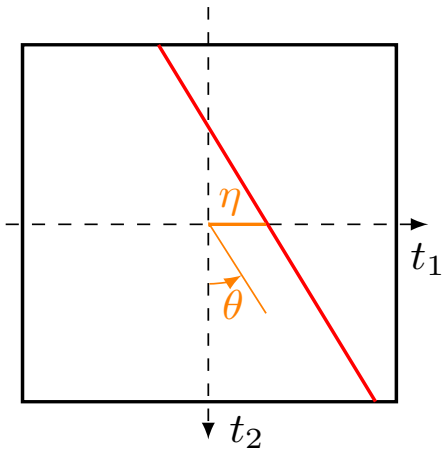
Reference : (Xu et al., 2013), (Chi and Chen, 2015)

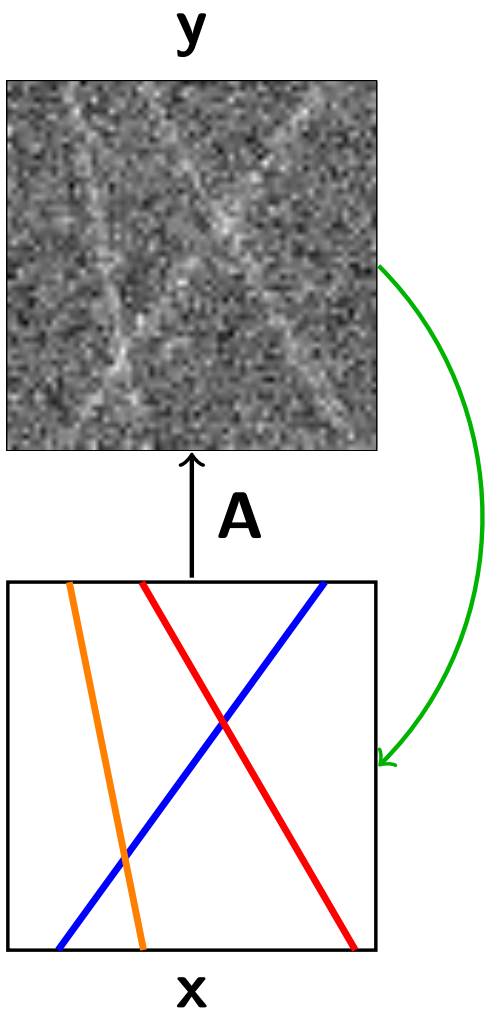


$$x = \sum_{k=1}^{K} c_k \delta_{(t_{k1}, t_{k2})}$$

Super-resolution of 2-D lines off-the-grid

$$x^\sharp(t_1, t_2) = \sum_{k=1}^{K} \alpha_k \delta\big(\cos\theta_k(t_1 - \eta_k) + \sin\theta_k\, t_2\big)$$





Super-resolution of 2-D lines off-the-grid

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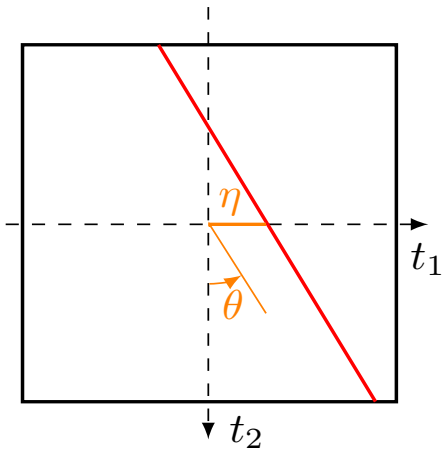
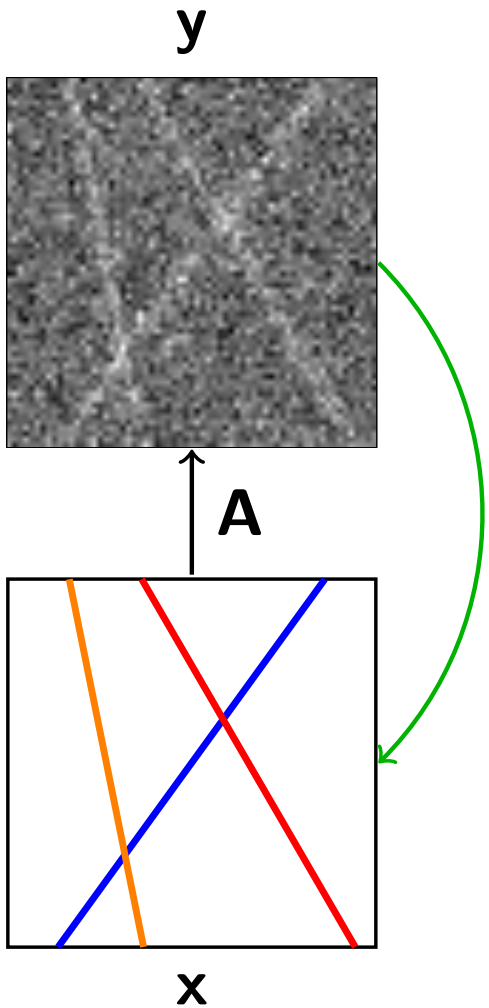


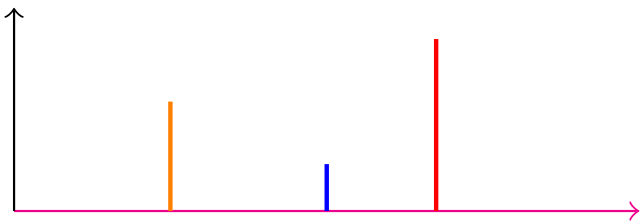


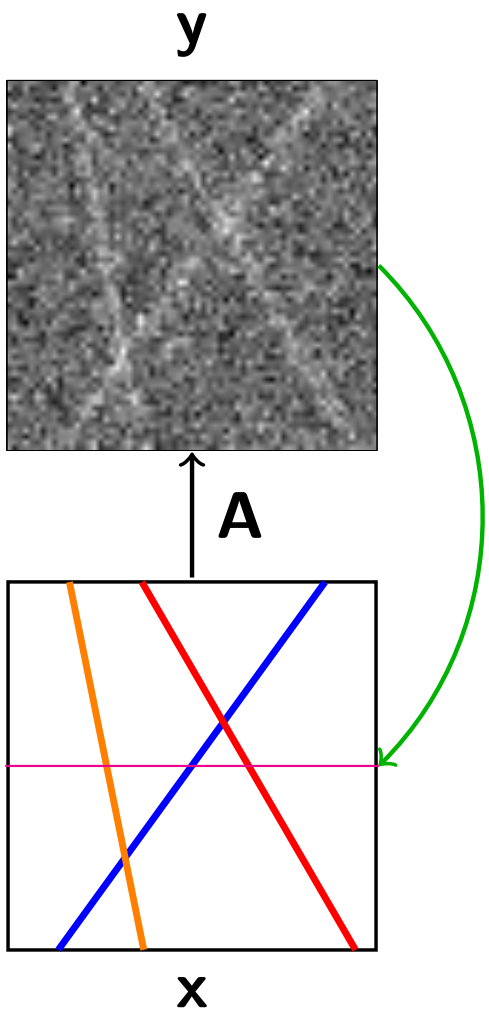
- ✓ Atomic formulation in Fourier?
- ✗ Compute atomic norm $\|\widehat{\mathbf{x}}^\sharp\|_{\mathcal{A}_{2D}}$?

  Super-resolution of 2-D lines?

Super-resolution of 2-D lines off-the-grid

$$x^{\sharp}_{t_2}(t_1) = \sum_{k=1}^{K} \alpha_k \delta\big(\cos\theta_k(t_1 - \eta_k) + \sin\theta_k\, t_2\big)$$

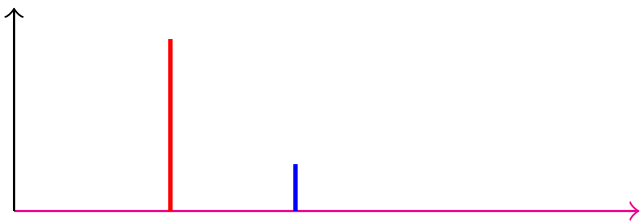




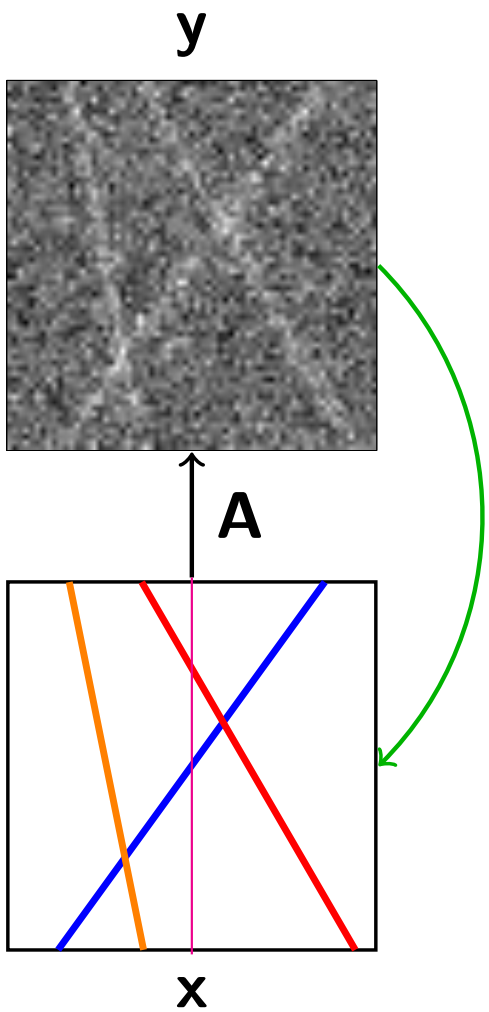
- ✓ Atomic formulation in Fourier?
- ✓ Compute atomic norm $\|\widehat{\mathbf{x}}^{\sharp}_{t_2}\|_{\mathcal{A}_{1D}}$?
- ✓ Super-resolution of 2-D lines?

Super-resolution of 2-D lines off-the-grid

$$x^{\sharp}_{t_1}(t_2) = \sum_{k=1}^{K} \alpha_k \delta\big(\cos\theta_k(t_1 - \eta_k) + \sin\theta_k\, \underline{t_2}\big)$$



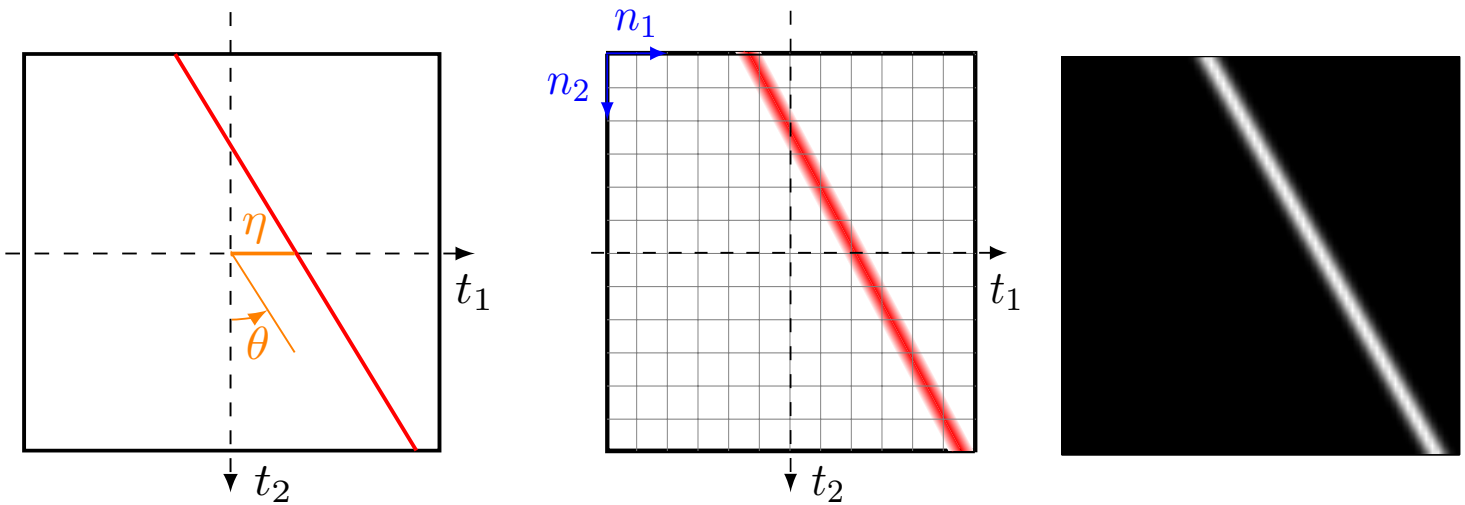
- ✓ Atomic formulation in Fourier?
- ✓ Compute atomic norm $\|\widehat{\mathbf{x}}^{\sharp}_{t_1}\|_{\mathcal{A}_{1\mathrm{D}}}$?
- ✓ Super-resolution of 2-D lines?



Modeling the blurred lines

$$x^\sharp : (t_1, t_2) \in \mathbb{P} \mapsto \sum_{k=1}^{K} \alpha_k \delta\big(\cos\theta_k(t_1 - \eta_k) + \sin\theta_k\, t_2\big)$$

$$\mathbf{b}^\sharp[n_1, n_2] = (x^\sharp * \phi)(n_1, n_2), \quad \phi(n_1, n_2) = \boldsymbol{g}[n_1]\boldsymbol{h}[n_2]$$



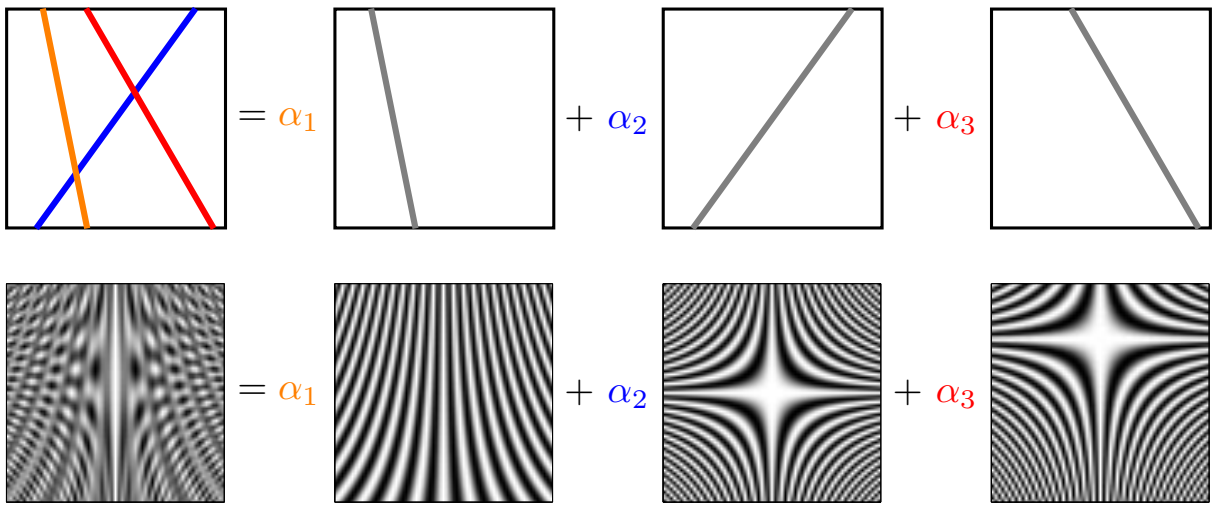
$x^\sharp$ $\quad$ $x^\sharp * \phi$ $\quad$ $\mathbf{b}^\sharp$

Modeling the blurred lines

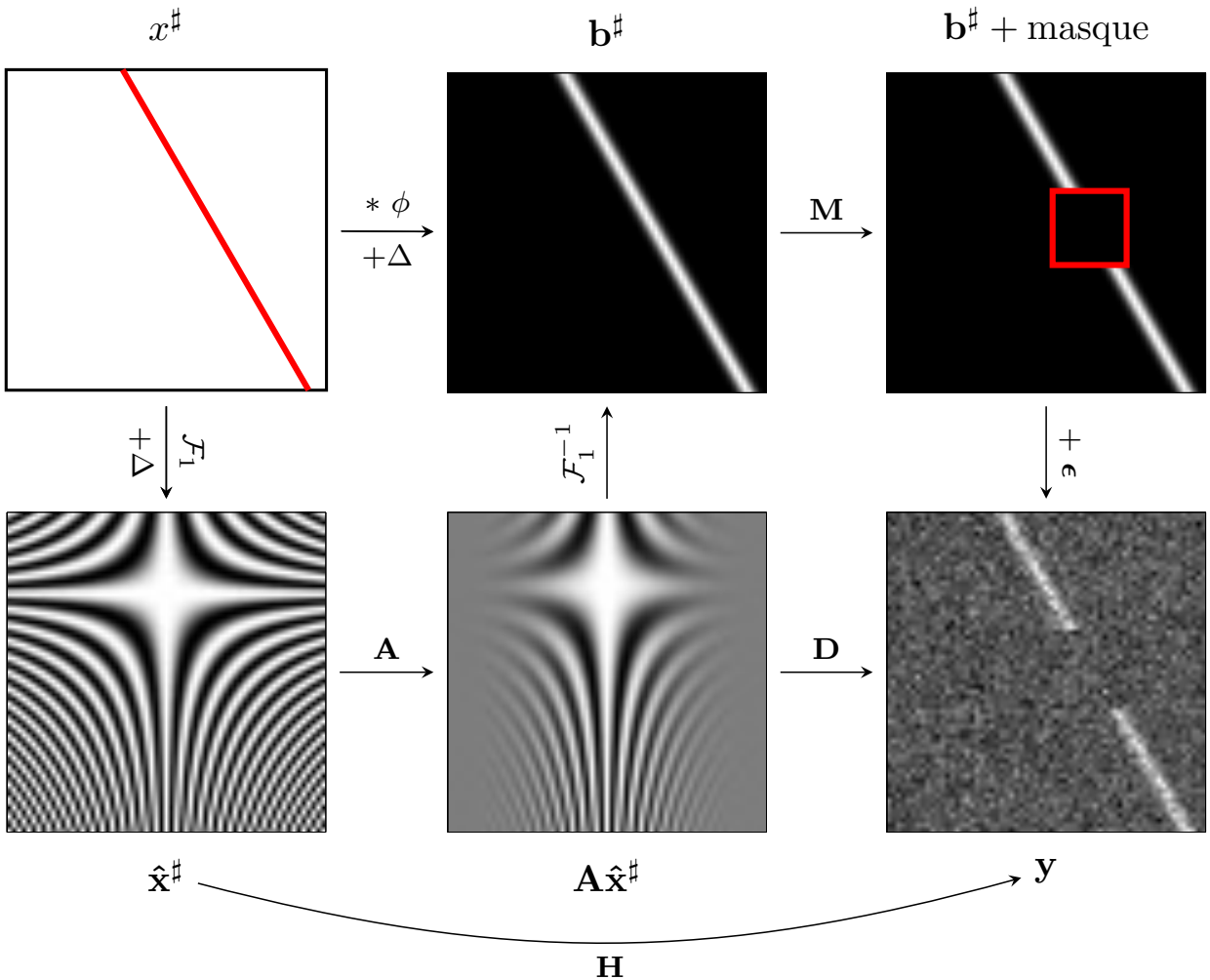
$$\widehat{\mathbf{x}}^{\sharp}[m, n_2] = (\mathcal{F}_1 x^{\sharp})[m, n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right)m}$$

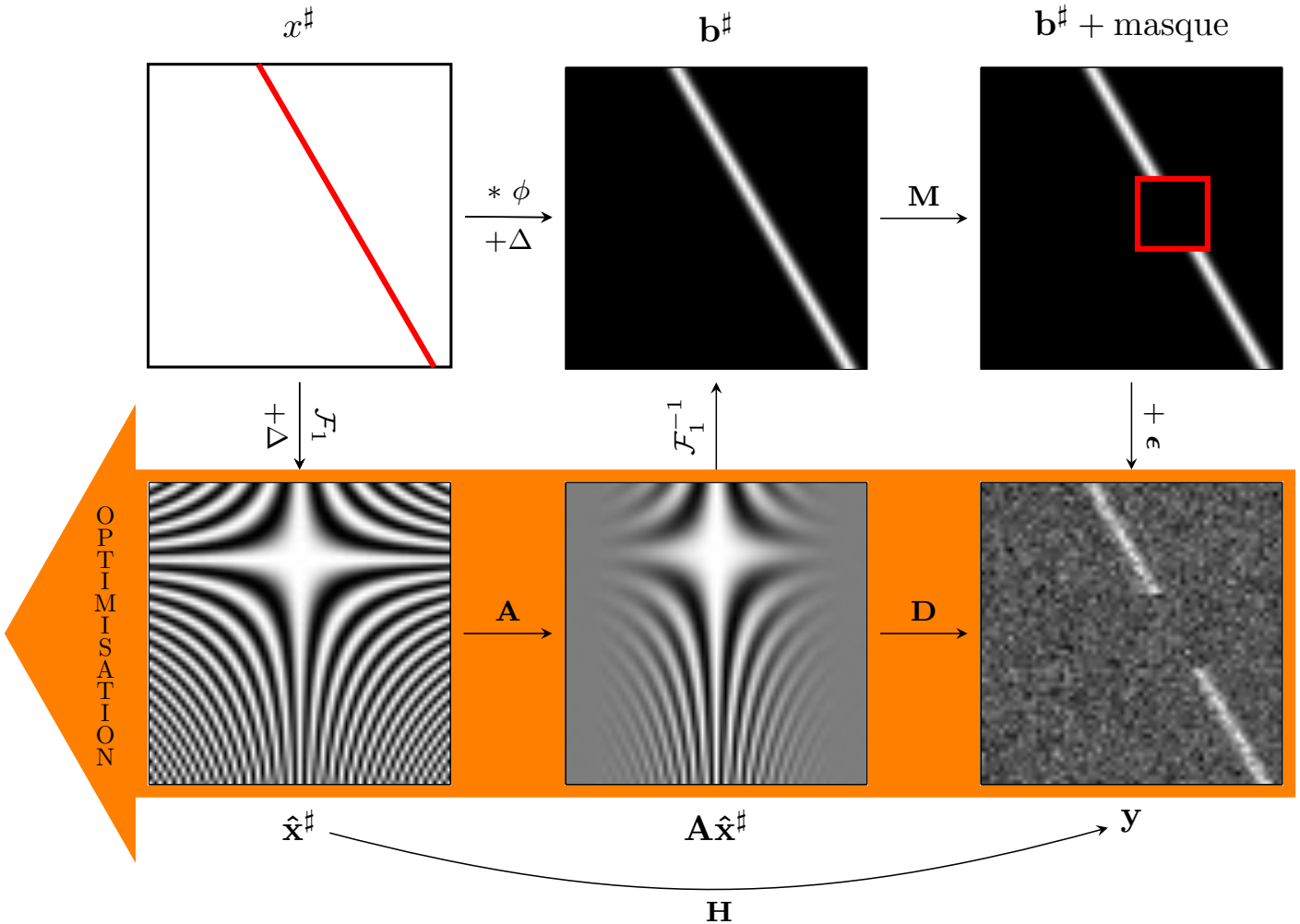
$$c_k = \frac{\alpha_k}{\cos\theta_k} \geq 0$$

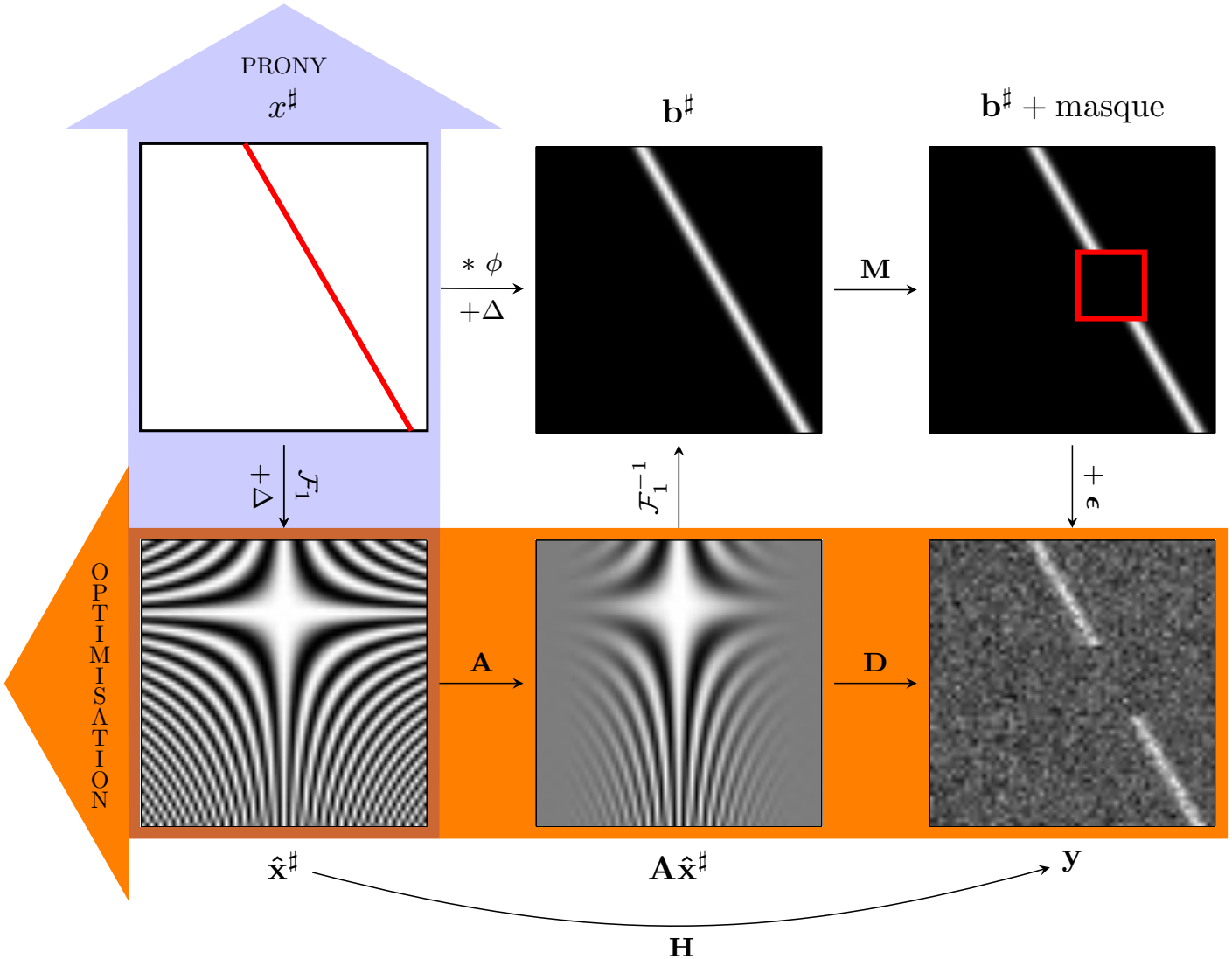
$$\widehat{\mathbf{b}}^{\sharp}[m, :] = (\widehat{\boldsymbol{g}}[m]\widehat{\mathbf{x}}^{\sharp}[m, :]) * \boldsymbol{h} \rightarrow \mathbf{A}\widehat{\mathbf{x}}^{\sharp} = \widehat{\mathbf{b}}^{\sharp}$$



Reconstruction steps




$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp + \text{masque}$
$* \phi$
$+\Delta$
$\mathbf{M}$
$\nabla +$
$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
$+\boldsymbol{\epsilon}$
OPTIMISATION
$\mathbf{A}$
$\mathbf{D}$
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
$\mathbf{H}$


PRONY
$x^\sharp$
$\mathbf{b}^\sharp$
$\mathbf{b}^\sharp$ + masque
$* \phi$
$+\Delta$
M
$\nabla +$
$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
$+ \epsilon$
OPTIMISATION
A
D
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
H

Atomic decomposition of columns and rows

$$\widehat{\mathbf{x}}^{\sharp}[\underline{m}, n_2] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right)m}, \quad m = -M, \ldots, M$$

$$\widehat{\mathbf{x}}^{\sharp}[m, \underline{n_2}] = \sum_{k=1}^{K} c_k \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} m\right)n_2 + \frac{2\pi\eta_k m}{W}}, \quad n_2 = 0, \ldots, H-1$$

1. $\boldsymbol{l}^{\sharp}_{n_2} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k}, \boxed{0})$ (columns of $\widehat{\mathbf{x}}$, without phase)
2. $\boldsymbol{t}^{\sharp}_{m} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \boxed{\phi_{m,k}})^{\mathsf{T}}$ (rows of $\widehat{\mathbf{x}}$, with phase)

$$[\boldsymbol{a}(f, \phi)]_i = \mathrm{e}^{\mathrm{j}(2\pi f i + \phi)} \in \mathcal{A}$$

Caratheodory theorem

Theorem (Caratheodory, 1907)

A vector $\boldsymbol{z} = (z_{N-1}^*, \ldots, z_1^*, z_0, z_1, \ldots, z_{N-1})$, *with* $z_0 \in \mathbb{R}$, *is a positive combination of* $K \leqslant N$ *atoms* $\boldsymbol{a}(f_k, \boxed{0})$ *if and only if* $\mathbf{T}_N(\boldsymbol{z}_+) \succcurlyeq 0$ *and is of rank* K, *where* $\boldsymbol{z}_+ = (z_0, \ldots, z_{N-1})$ *and*

$$\mathbf{T}_N : \boldsymbol{z}_+ = (z_0, \ldots, z_{N-1}) \mapsto \begin{pmatrix} z_0 & z_1^* & \cdots & z_{N-1}^* \\ z_1 & z_0 & \cdots & z_{N-2}^* \\ \vdots & \vdots & \ddots & \vdots \\ z_{N-1} & z_{N-2} & \cdots & z_0 \end{pmatrix}.$$

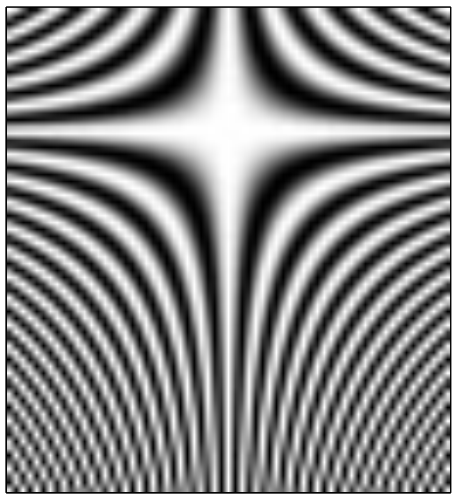
Moreover, this decomposition is unique if $K < N$.

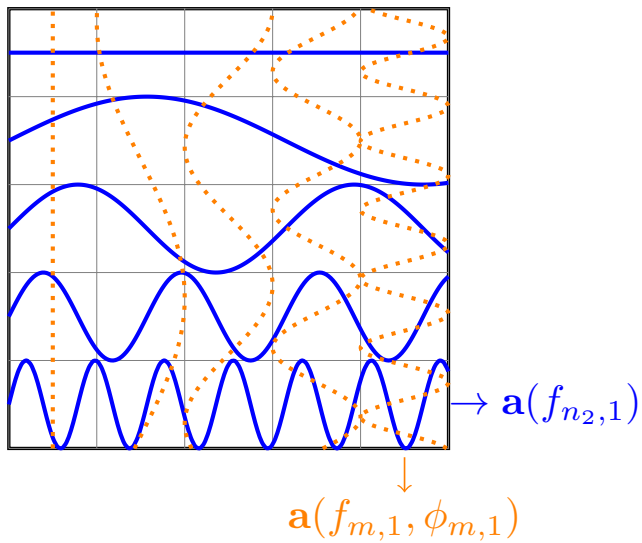
Atomic decomposition of one line ($K = 1$)

$$\widehat{\mathbf{x}}^{\sharp}[m, n_2] = c_1 \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_1}{W} n_2 + \frac{\eta_1}{W}\right)m} = c_1 \mathrm{e}^{\mathrm{j}2\pi((f_1 - f_0)n_2 + f_0)m}$$

with $f_0 = -\eta_1/W$ and $f_1 = (\tan\theta_1 - \eta_1)/W$.

1. $\boldsymbol{l}^{\sharp}_{n_2} = c_1 \boldsymbol{a}(f_{n_2,1}, 0)$ (one atom without phase)
2. $\boldsymbol{t}^{\sharp}_{m} = c_1 \boldsymbol{a}(f_{m,1}, \phi_{m,1})^{\mathsf{T}}$ (one atom with phase)





Atomic characterization of one line ($K = 1$)

Characterization of one sampled line in Fourier

A matrix $\widehat{\mathbf{x}}$ is of the form $\widehat{\mathbf{x}}[m,n] = c_1 \mathrm{e}^{\mathrm{j}2\pi((f_1-f_0)n+f_0)m)}$ if and only if the columns $\boldsymbol{l}_n$ and rows $\boldsymbol{t}_m$ of $\widehat{\mathbf{x}}$ are such that $\mathbf{T}_M(\boldsymbol{l}_n) \succcurlyeq 0$ and of rank 1, $\mathbf{P}_1(\boldsymbol{t}_m)$ is of rank 1 and $\widehat{\mathbf{x}}[0,n] = \widehat{\mathbf{x}}[0,0]$ for all m and n.

$$\underbrace{\begin{pmatrix} z_0 & z_1^* & \cdots & z_{N-1}^* \\ z_1 & z_0 & \cdots & z_{N-2}^* \\ \vdots & \vdots & \ddots & \vdots \\ z_{N-1} & z_{N-2} & \cdots & z_0 \end{pmatrix}}_{=\mathbf{T}_N(\boldsymbol{z})} \quad \underbrace{\begin{pmatrix} z_K & z_{K-1} & \cdots & z_0 \\ z_{K+1} & z_K & \cdots & z_1 \\ \vdots & \vdots & \ddots & \vdots \\ z_{N-1} & z_{N-2} & \cdots & z_{N-K-1} \end{pmatrix}}_{=\mathbf{P}_K(\boldsymbol{z})}.$$

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Convex relaxation: since $\mathbf{T}_M(\boldsymbol{l}_{n_2}) = \mathbf{V}_{n_2} \mathrm{diag}(c_1, \ldots, c_K) \mathbf{V}_{n_2}^*$ with the Vandermonde matrix $\mathbf{V}_{n_2} = (\boldsymbol{a}(f_{n_2,1}), \ldots, \boldsymbol{a}(f_{n_2,K}))$, we get:

$$\|\mathbf{T}_M(\boldsymbol{l}_{n_2})\|_* \propto \|\boldsymbol{l}_{n_2}\|_{\mathcal{A}}$$

Atomic norms computation

1. $I^{\sharp}_{n_2} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k}, \boxed{0}) \Leftrightarrow \mathbf{T}_{M+1}(I^{\sharp}_{n_2}) \succcurlyeq 0 +$ of rank K

Atomic norm without phase (Caratheodory, 1907)

Since $K < M$ the decomposition is unique then

$$\|I^{\sharp}_{n_2}\|_{\mathcal{A}} = \sum_{k=1}^{K} c_k = \widehat{\mathbf{x}}^{\sharp}[0, n_2] = c^{\star}$$

2. $t^{\sharp}_m = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \boxed{\phi_{m,k}})^{\mathsf{T}}$

Atomic norm with phase (Tang et al., 2013)

$$\|t^{\sharp}_m\|_{\mathcal{A}} = \inf_{\boldsymbol{q}\in\mathbb{C}^H, t\in\mathbb{R}} \left\{ \frac{1}{2}\mathrm{Tr}(\mathbf{T}_N(\boldsymbol{q})) + \frac{1}{2}t : \begin{pmatrix} \mathbf{T}_N(\boldsymbol{q}) & t^{\sharp}_m \\ t^{\sharp}_m{}^{*} & t \end{pmatrix} \succcurlyeq 0 \right\}$$

Atomic norms computation

1. $I^{\sharp}_{n_2} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k}, \boxed{0}) \Leftrightarrow \mathbf{T}_{M+1}(I^{\sharp}_{n_2}) \succcurlyeq 0 +$ of rank K

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2. $t^{\sharp}_m = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \boxed{\phi_{m,k}})^{\mathsf{T}}$

Atomic norm with phase

$$\|t^{\sharp}_m\|_{\mathcal{A}} = \min_{\boldsymbol{q} \in \mathbb{C}^H} \left\{ q_0 : \underbrace{\begin{pmatrix} \mathbf{T}_N(\boldsymbol{q}) & t^{\sharp}_m \\ t^{\sharp}_m{}^{*} & q_0 \end{pmatrix}}_{\mathbf{T}'_N(t^{\sharp}_m, \boldsymbol{q})} \succcurlyeq 0 \right\} \equiv \underbrace{\mathrm{SDP}(t^{\sharp}_m)}_{\boldsymbol{q}_m[0]} \leqslant c^{\star}$$

Convex optimization problem

Proposition (Convex minimization)

$$\tilde{\mathbf{x}} \in \underset{\widehat{\mathbf{x}},\mathbf{q}\in\mathcal{X}\times\mathcal{Q}}{\arg\min} \frac{1}{2}\|\mathbf{A}\widehat{\mathbf{x}} - \widehat{\mathbf{y}}\|^2 \ ,$$

$$\text{under constraints} \quad \begin{cases} \forall n_2 = 0, ..., H-1, \ \forall m = 1, ..., M \ , \\ \widehat{\mathbf{x}}[0, n_2] = \widehat{\mathbf{x}}[0, 0] \leqslant c \ , \\ \mathbf{q}[m, 0] \leqslant c \ , \\ \mathbf{T}'_H(\widehat{\mathbf{x}}[m, :], \mathbf{q}[m, :]) \succcurlyeq 0 \ , \\ \mathbf{T}_{M+1}(\widehat{\mathbf{x}}[:, n_2]) \succcurlyeq 0 \ . \end{cases}$$

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(Chambolle et Pock, 2010)

$$\tilde{\mathbf{X}} = \underset{\mathbf{X}\in\mathcal{H}}{\arg\min} \left\{ F(\mathbf{X}) + G(\mathbf{X}) + \sum_{i=0}^{Q-1} H_i(\mathrm{L}_i(\mathbf{X})) \right\}$$

Convex optimization problem

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$$\tilde{\mathbf{x}} \in \underset{\widehat{\mathbf{x}},\mathbf{q}\in\mathcal{X}\times\mathcal{Q}}{\arg\min} \frac{1}{2}\|\mathbf{A}\widehat{\mathbf{x}} - \widehat{\mathbf{y}}\|^2 \ ,$$

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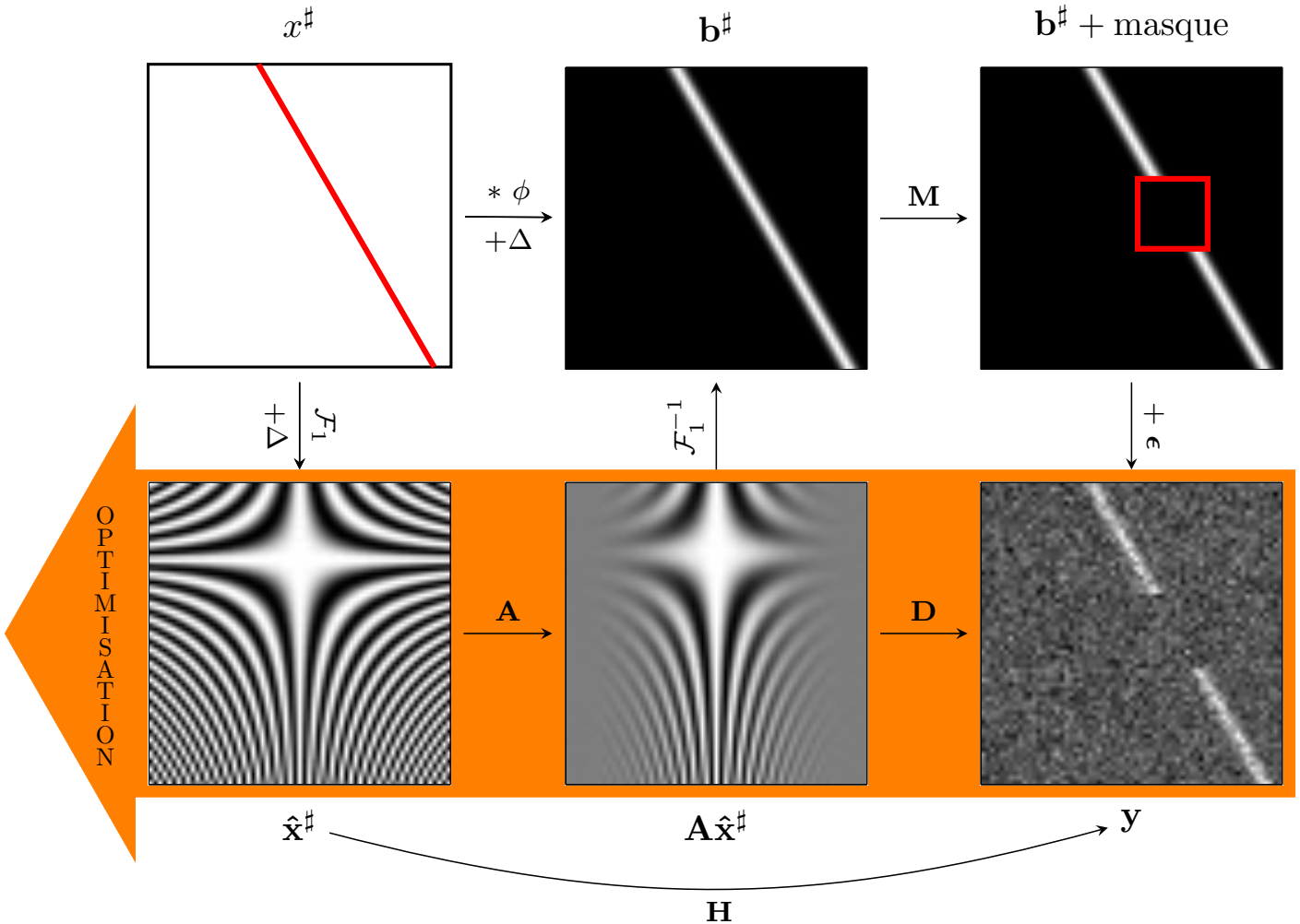
(Chambolle and Pock, 2010)

$$\tilde{\mathbf{X}} = \underset{\mathbf{X}\in\mathcal{H}}{\arg\min} \left\{ F(\mathbf{X}) + G(\mathbf{X}) + \sum_{i=0}^{Q-1} H_i(\mathrm{L}_i(\mathbf{X})) \right\}$$


$x^\sharp$
$\mathbf{b}^\sharp$
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$* \phi$
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M
$\nabla+$
$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
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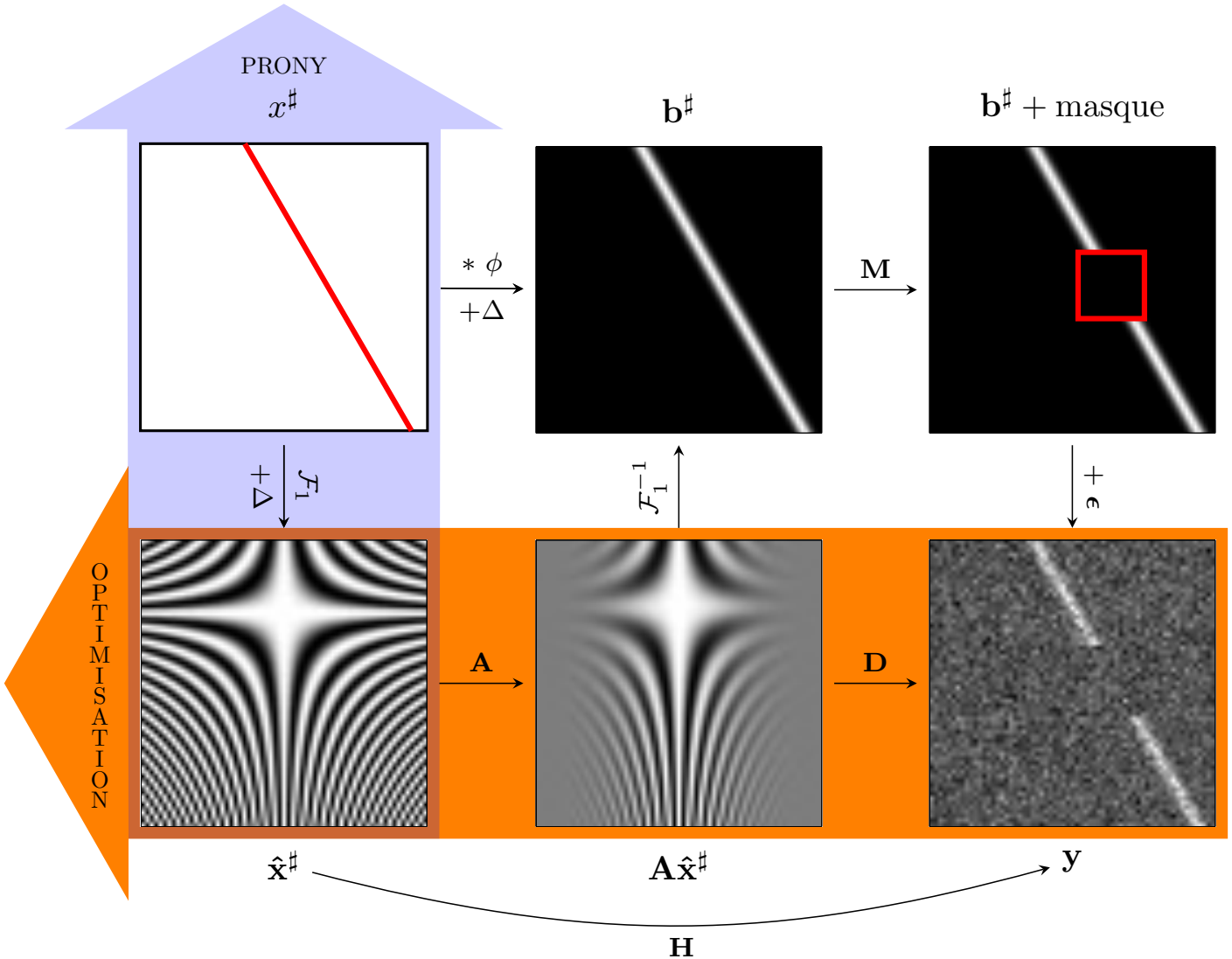
OPTIMISATION
A
D
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
H


PRONY
$x^\sharp$
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$\mathcal{F}_1$
$\mathcal{F}_1^{-1}$
$+ \epsilon$
OPTIMISATION
A
D
$\hat{\mathbf{x}}^\sharp$
$\mathbf{A}\hat{\mathbf{x}}^\sharp$
$\mathbf{y}$
H

Super-resolution of lines in two steps

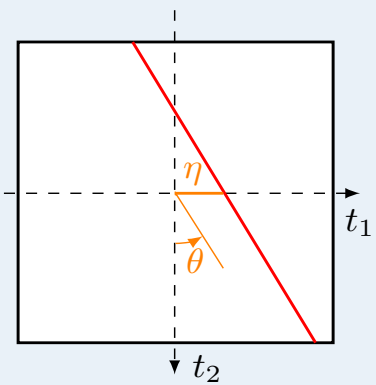
Convex optimization $\mathbf{y} \rightarrow \widehat{\mathbf{x}}$

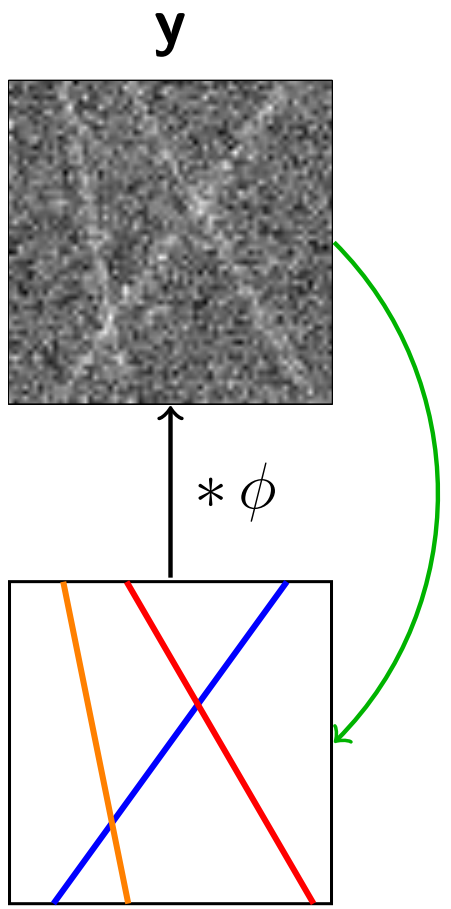
$$\arg\min_{\widehat{\mathbf{x}}} \frac{1}{2} \left\| \widehat{\mathbf{y}} - \mathbf{A}\widehat{\mathbf{x}} \right\|^2 + \lambda \sum_n \left\| \widehat{\mathbf{x}}_n \right\|_{\mathcal{A}_{1D}}$$

Spectral estimation $\widehat{\mathbf{x}} \rightarrow \{\alpha_k, \theta_k, \eta_k\}_k$

$$\widehat{\mathbf{x}}^\sharp[m, n_2] = \sum_{k=1}^{K} \frac{\alpha_k}{\cos\theta_k} \mathrm{e}^{\mathrm{j}2\pi\left(\frac{\tan\theta_k}{W} n_2 + \frac{\eta_k}{W}\right)m}$$

Prony





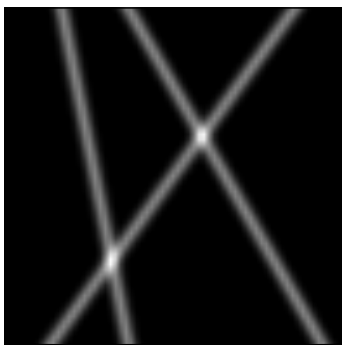
$$\{\alpha_k, \theta_k, \eta_k\}_{k=1}^K \rightarrow x^\sharp$$

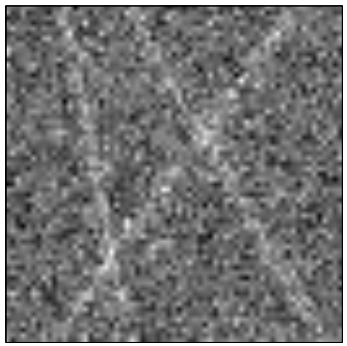
Numerical experiments

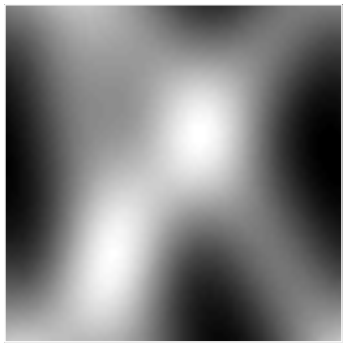
- Denoising lines:

Numerical experiments

- Denoising and deconvolution


Exp. 1


Exp. 2


Exp 3.

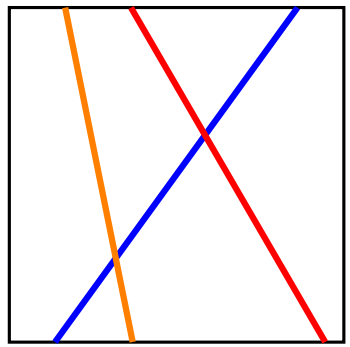

Détection

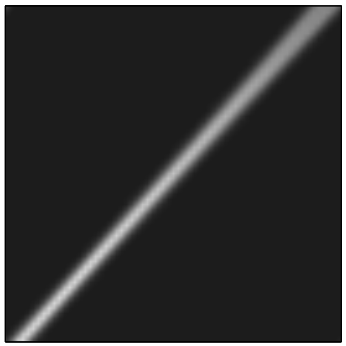
Table: Relative errors of the line parameters estimation

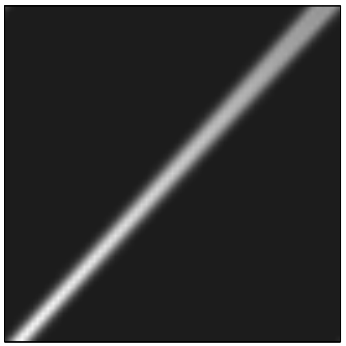
	Experiment 1	**Experiment 2**	**Experiment 3**
Δ_θ/θ	$(10^{-7}, 3.10^{-6}, 7.10^{-7})$	$(10^{-2}, 6.10^{-2}, 9.10^{-2})$	$(6.10^{-7}, 9.10^{-5}, 8.10^{-6})$
Δ_α/α	$(10^{-7}, 10^{-7}, 10^{-7})$	$(10^{-2}, 9.10^{-2}, 2.10^{-1})$	$(4.10^{-5}, 2.10^{-5}, 2.10^{-5})$
Δ_η	$(4.10^{-6}, 7.10^{-6}, 7.10^{-6})$	$(5.10^{-2}, 4.10^{-2}, 3.10^{-2})$	$(5.10^{-5}, 10^{-4}, 3.10^{-4})$

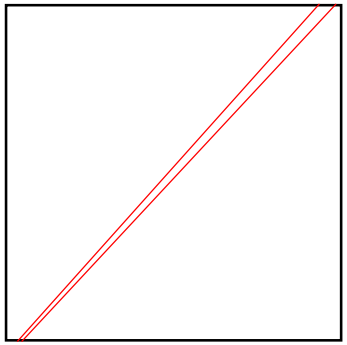
Numerical experiments

- Closed lines

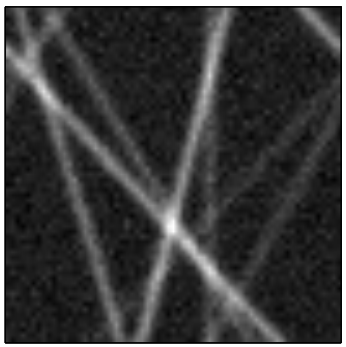
Noisy

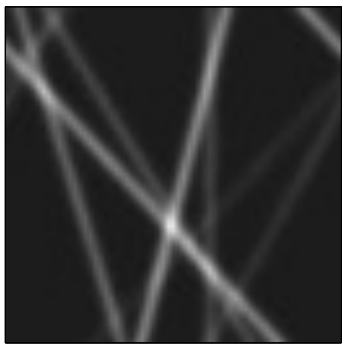

Noisy

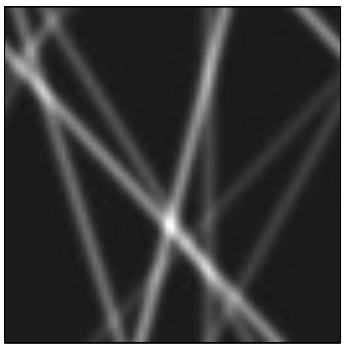

Ground truth

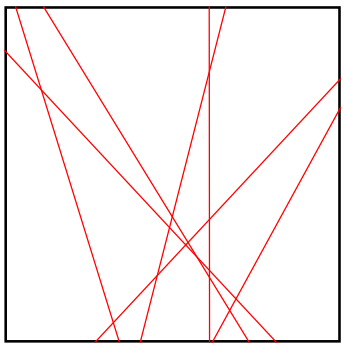

Detection

- Multiple lines


Noisy

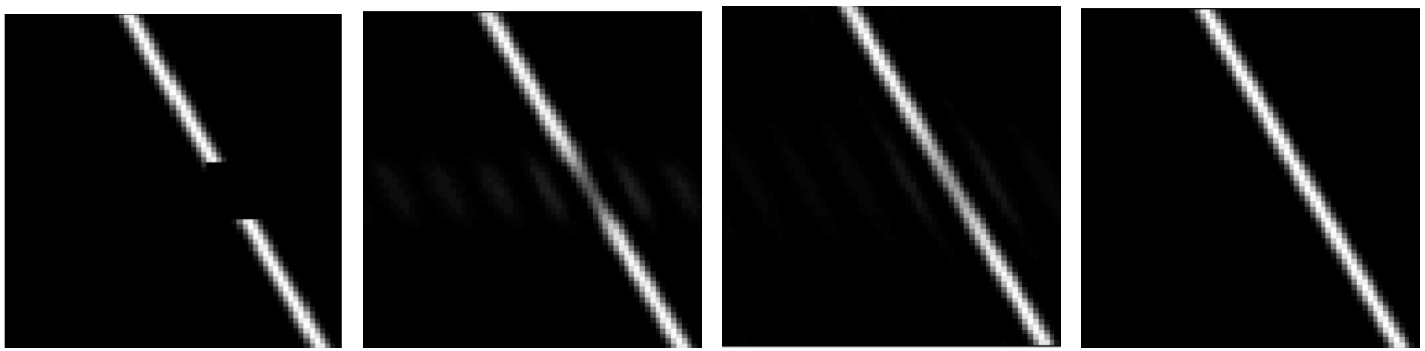

Noisy


Ground truthtt


Detection

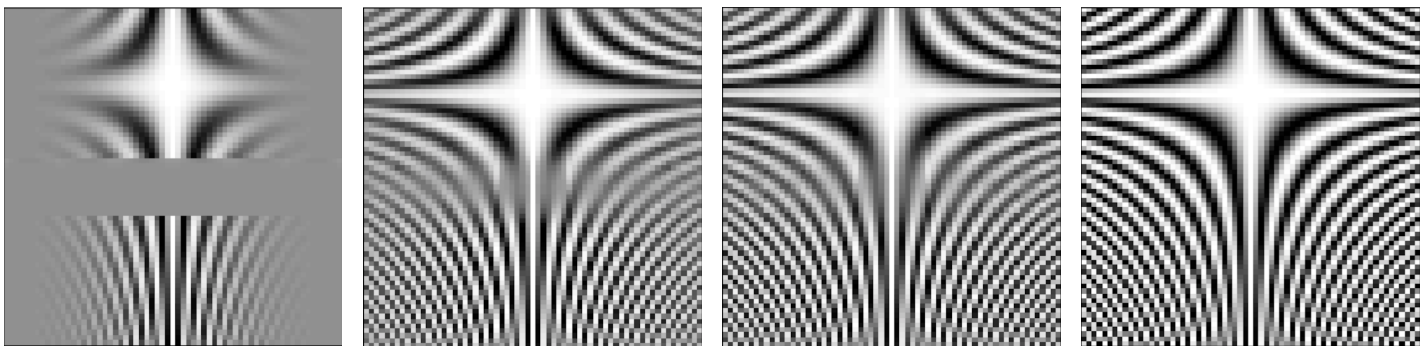
Numerical experiments

- Spatial inpainting



Masking iter = 2000 iter = 10000 iter $\to \infty$

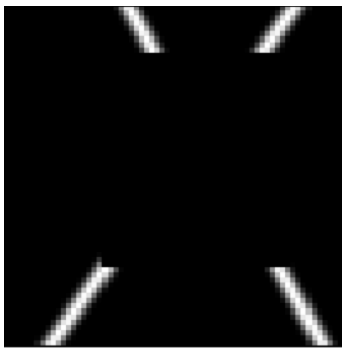
- Inpainting in Fourier

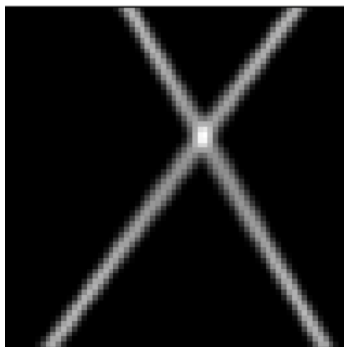


Masquage iter = 2000 iter = 10000 iter $\to \infty$

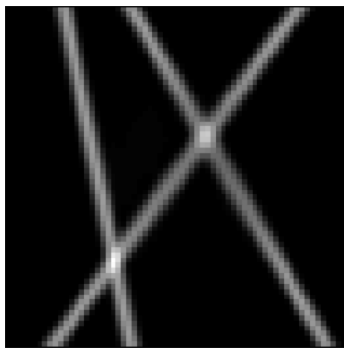
Numerical experiments

- Inpainting with big mask


Masquage

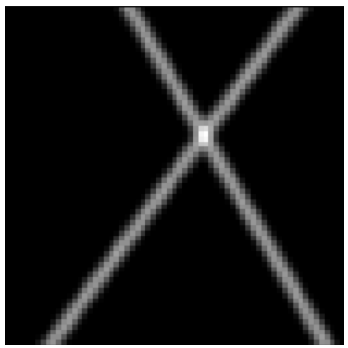

Inpainting

Masquage

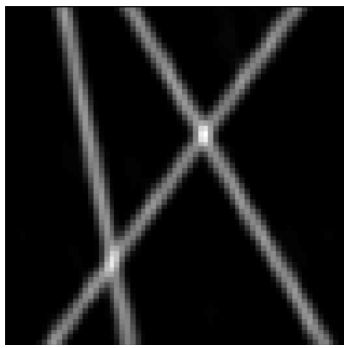

Inpainting

- Inpainting with random mask

Masquage


Inpainting

Masquage


Inpainting

Conclusion

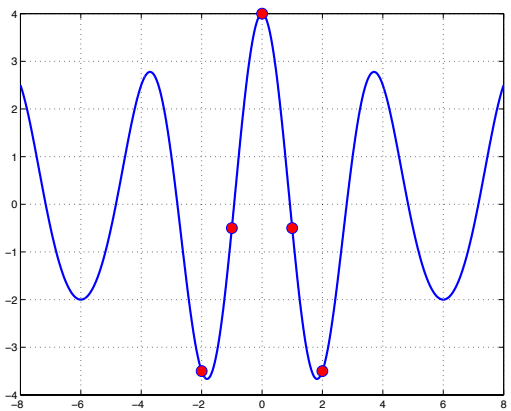
- ✓ New method for the super-resolution of 2-D lines, with dedicated 1-D atomic norms penalities enforcing sparsity
- ✓ Penalize in both directions can lead to the exact solution
- ✗ No theoretical guarantees about separation conditions and statistical analysis for recovering the exact solution
- ✓ Solving the convex optimization problem by a primal-dual splitting algorithm
- ✗ Slow convergence : for each iteration perform SVD onto all the Toeplitz matrices made from rows and columns
- ✓ Extraction of the line parameters combining spectral estimation on rows/columns achieves subpixel accuracy

Questions?

Thank you for your attention

Frequencies extraction from columns and rows

1. $l^{\sharp}_{n_2} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{n_2,k}, 0)$ (columns of $\widehat{\mathbf{x}}$, without phase)
2. $t^{\sharp}_{m} = \sum_{k=1}^{K} c_k \boldsymbol{a}(f_{m,k}, \phi_{m,k})^{\mathsf{T}}$ (rows de $\widehat{\mathbf{x}}$, with phase)



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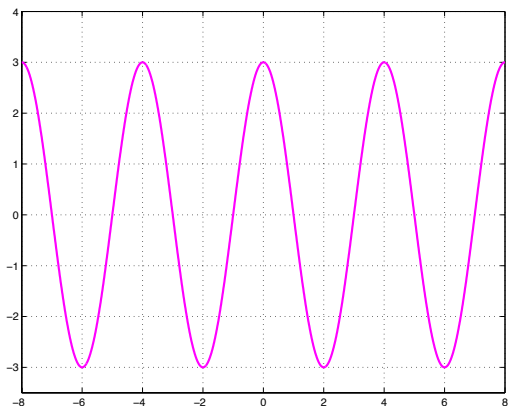

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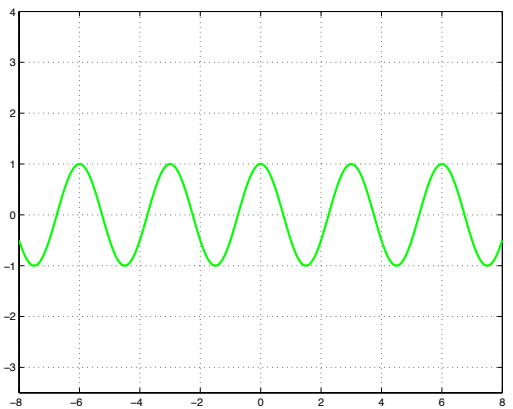


$x(t) = x_1(t) + x_2(t)$ $\quad x_1(t) = \boxed{3} \exp\left(\mathrm{j}2\pi \boxed{\frac{1}{4}} t\right)$ $\quad x_2(t) = \boxed{1} \exp\left(\mathrm{j}2\pi \boxed{\frac{1}{3}} t\right)$

$\Rightarrow$ spectral method estimation (Prony, ESPRIT, MUSIC, Matrix Pencil...)

Prony method

$$x_m = \sum_{k=1}^{K} \rho_k \underbrace{\left(e^{-j\omega_k}\right)}_{z_k}^{m}, \quad \rho_k \in \mathbb{C},\ \omega_k \in [-\pi, \pi],\ m = -M, \ldots, M$$

$$\text{Annihilating filter : } H(z) = \prod_{k=1}^{K}(z - \overline{z_k}) = \sum_{k=0}^{K} h_k z^k$$

$$\sum_{j=0}^{K} h_j x_{m-j} = \sum_{j=0}^{K} h_j \left(\sum_{k=1}^{K} \rho_k z_k^{m-j} \right) = \sum_{k=1}^{K} \rho_k z_k^m \underbrace{\left(\sum_{j=0}^{K} h_j z_k^{-j} \right)}_{H(\overline{z_k})=0} = 0$$

Prony method : annihilating polynomial

$$x_m = \sum_{k=1}^{K} \rho_k \underbrace{\left(e^{-j\omega_k}\right)}_{z_k}^{m}, \quad \rho_k \in \mathbb{C},\ \omega_k \in [-\pi, \pi],\ m = -M, \ldots, M$$

Annihilating filter : $H(z) = \prod_{k=1}^{K} (z - \overline{z_k}) = \sum_{k=0}^{K} h_k z^k$

- $\sum_{j=0}^{K} h_j x_{m-j} = 0,\ \forall m = -M+K, \ldots, M \Leftrightarrow \boldsymbol{x} * \boldsymbol{h} = \mathbf{0}$
- $\begin{pmatrix} x_{-M+K} & \cdots & x_{-M} \\ \vdots & \ddots & \vdots \\ x_M & \cdots & x_{M-K} \end{pmatrix} \begin{pmatrix} h_0 \\ \vdots \\ h_K \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} \Leftrightarrow \mathbf{T}_K \boldsymbol{h} = \mathbf{0}$

Prony method : frequencies estimation

$$x_m = \sum_{k=1}^{K} \rho_k \underbrace{\left(e^{-j\omega_k}\right)}_{z_k}^{m}, \quad \rho_k \in \mathbb{C},\ \omega_k \in [-\pi, \pi],\ m = -M, \ldots, M$$

Annihilating filter: $H(z) = \prod_{k=1}^{K}(z - \overline{z_k}) = \sum_{k=0}^{K} h_k z^k$

- $\boldsymbol{h}$ = sing. vec. for $\lambda = 0$ of

$$\mathbf{T}_K = \begin{pmatrix} x_{-M+K} & \cdots & x_{-M} \\ \vdots & \ddots & \vdots \\ x_M & \cdots & x_{M-K} \end{pmatrix}$$

- $\overline{z_k}$ = roots of the polynomial $H(z)$, puis $\omega_k = \arg(\overline{z_k})$

Prony method : amplitudes estimation

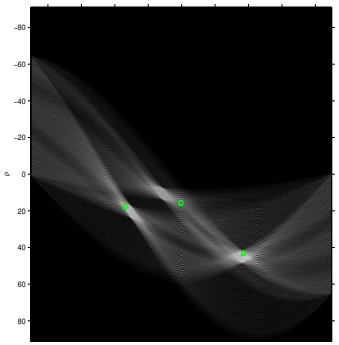
- $x_m = \sum_{k=1}^{K} \rho_k \left(e^{-j\omega_k}\right)^m, \forall m = -M, \ldots, M$
- $\begin{pmatrix} e^{jM\omega_1} & \ldots & e^{jM\omega_K} \\ \vdots & \ddots & \vdots \\ e^{-jM\omega_1} & \ldots & e^{-jM\omega_K} \end{pmatrix} \begin{pmatrix} \rho_1 \\ \vdots \\ \rho_K \end{pmatrix} = \begin{pmatrix} x_{-M} \\ \vdots \\ x_M \end{pmatrix} \Leftrightarrow \mathbf{U}\boldsymbol{\rho} = \boldsymbol{x}$

Least-square method :

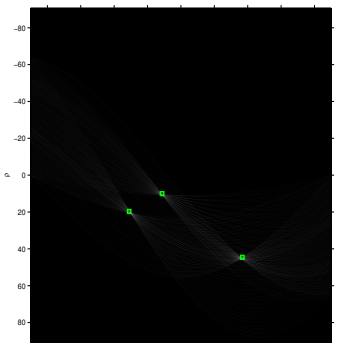
$$\mathbf{U}^{\mathrm{H}}\mathbf{U}\boldsymbol{\rho} = \mathbf{U}^{\mathrm{H}}\boldsymbol{x} \Longleftrightarrow \boldsymbol{\rho} = (\mathbf{U}^{\mathrm{H}}\mathbf{U})^{-1}\mathbf{U}^{\mathrm{H}}\boldsymbol{x}$$

Expériences numériques
Avec la transformée de Hough

- Espace de Hough pour la détection de lignes :



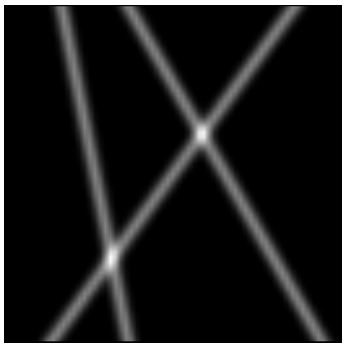
Squelettisation Détection pics Reconstruction



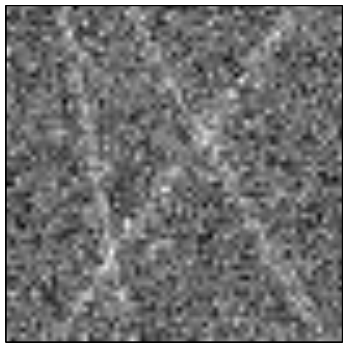
Squelettisation Détection pics Reconstruction

Expériences numériques
Avec la transformée de Radon

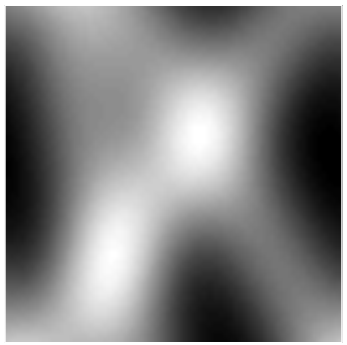
- Espace de Radon pour la détection de lignes :



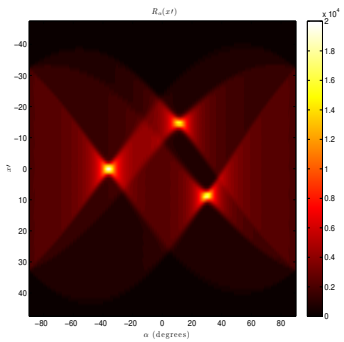
Exp. 1



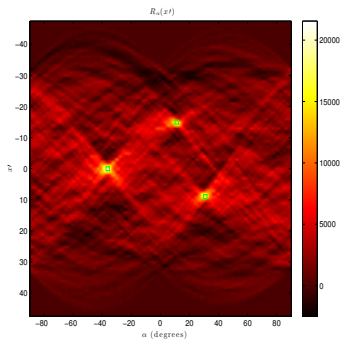
Exp. 2



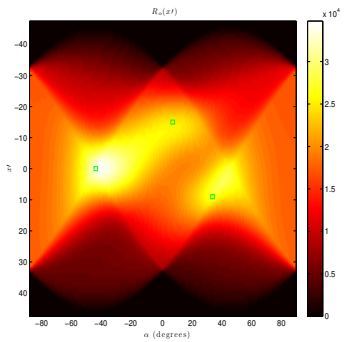
Exp 3.



Détection pics



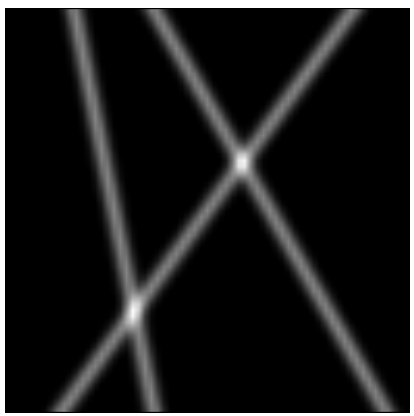
Détection pics

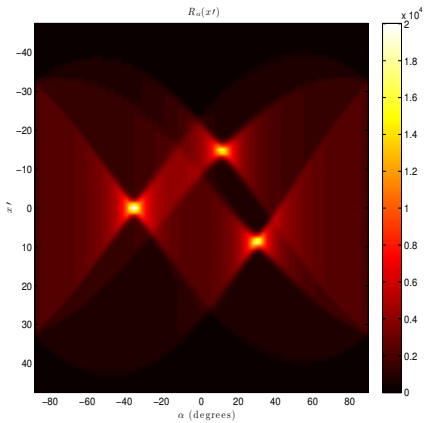


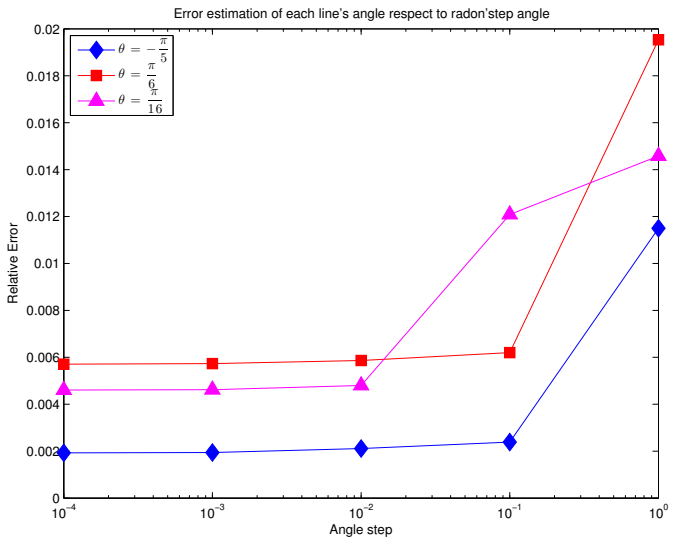
Détection pics

Expériences numériques
Avec la transformée de Radon

- Limite de ces transformées discrètes attachées à la grille :







Exp. 1